Local vulnerabilities and global robustness of equilibrium in network-coupled systems

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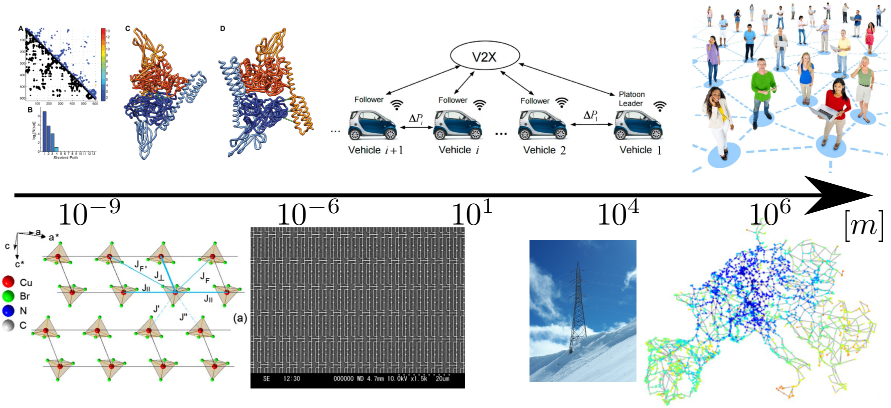
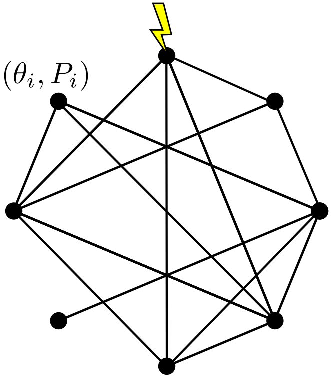
Introduction – Coupled Dynamical Systems



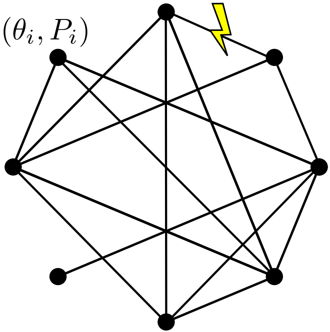
Sources: Phys. Rev. B **80**, 094411 (2009)
PLoS Comput Biol 11(6), e1004262 (2015)
Phys. Rev. Lett. **85**, 1974 (2000).
Transportation Research Part C: Emerging Technologies Volume 84, November 2017, Pages 21-47.
https://etranselec.ch/2019/10/07/Swissgri.html
https://researchoutreach.org/articles/opinion-dynamics-consensus-social-networks/

Perturbations in complex networks

Node

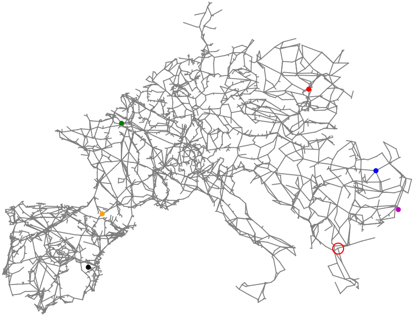


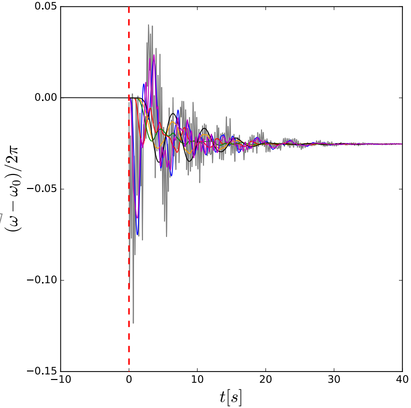
Edge



Introduction – Vulnerabilities and Robustness

Small perturbation $\rightarrow$ Transient dynamics





MT,Coletta, Jacquod, Phys. Rev. Lett. **120**, 084101 (2018).
MT,Pagnier, Jacquod Sci. Adv. **5**(11), eaaw8359 (2019).
MT, Jacquod, Phys. Rev. E **100**, 032303 (2019).
MT, Jacquod IEEE L-CSS **5**(3), 929-934 (2020).

Multistability → Transitions between fixed points

.. Silk, J. & Vilenkin, A. *Phys. Rev. Lett.* **53,** 1700 (1984).
9. Hogan, C. J. *Phys. Lett.* **143B,** 87 (1984).
10. Vilenkin, A. & Field, G. *Nature* (in the press).

Flux quantization in a high-T_c superconductor

C. E. Gough*, M. S. Colclough*, E. M. Forgan*, R. G. Jordan*, M. Keene*, C. M. Muirhead*, A. I. M. Rae*, N. Thomas*, J. S. Abell† & S. Sutton†

* Department of Physics, University of Birmingham, Birmingham B15 2TT, UK
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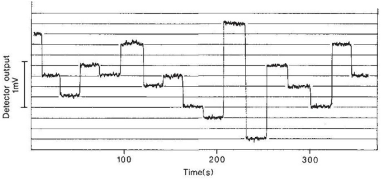
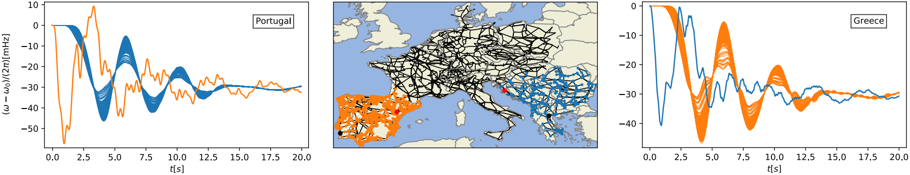


Fig. 2 Output of the r.f.-SQUID magnetometer showing small integral numbers of flux quanta jumping in and out of the ring.

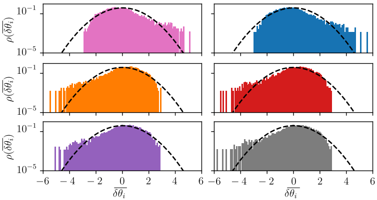
way we deduce that the magnitude of the flux jumps shown in Fig. 1 is typically 100 $(h/2e)$.

Delabays, MT, Jacquod, Chaos **27**(10), 103109 (2017).
MT, Delabays, Jacquod, Phys. Rev. E **99**(6), 062213 (2019).
Baumann, Sokolov, MT, Phys. Rev. E **102**(5), 052313 (2020).

Disturbance spreading



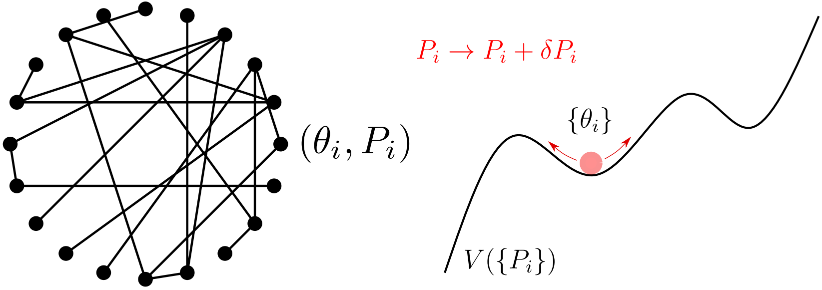
Non-Gaussian noise propagation



Fritzsch, MT, Jacquod proceedings 60th IEEE CDC (2021).
MT, Hindes, Jacquod, arXiv:2203.00590 (2022).

Introduction

Small perturbation response



Second order Kuramoto model

$$m_i\ddot{\theta}_i + d_i\dot{\theta}_i = P_i - \sum_j a_{ij}\sin(\theta_i - \theta_j) \ \ , \ \ i = 1, ..., n.$$

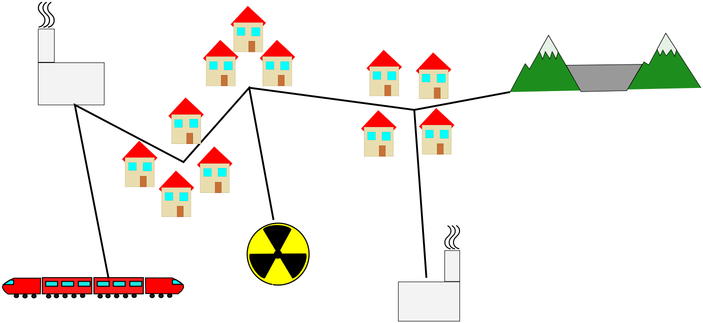
$a_{ij} = a_{ji} \geq 0$.
P_i : natural frequencies.
m_i : inertia.
d_i : damping.

Electric Power Network (in the lossless line approximation)

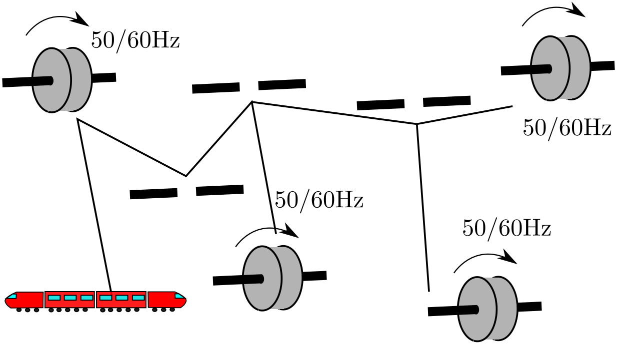
P_i : injected/consumed power.
$m_i = 0$: loads.
$m_i \neq 0$: generators.
$a_{ij}\sin(\theta_i - \theta_j)$: power flow from i to j .

J. A. Acebrón, L. L. Bonilla, Conrad J. Pérez Vicente, F. Ritort, and R. Spigler, Rev. Mod. Phys. **77**, 137 (2005)

Power grids



Power grids



Coupled Oscillators

Second order Kuramoto model

$$m_i\ddot{\theta}_i + d_i\dot{\theta}_i = P_i - \sum_j a_{ij}\sin(\theta_i - \theta_j) \ \ , \ i = 1, ..., n.$$

$a_{ij} = a_{ji} \geq 0$.

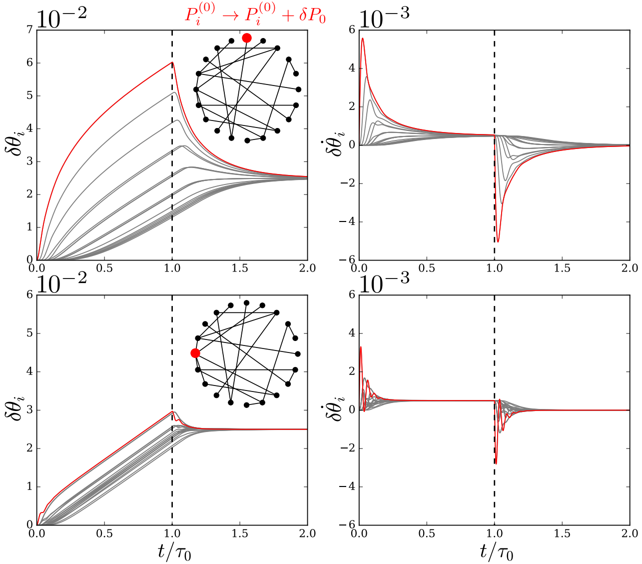
Steady-state solutions Synchronous states $\{\theta_i^{(0)}\}$ such that:

$$P_i = \sum_j a_{ij}\sin(\theta_i^{(0)} - \theta_j^{(0)}) \ \ , \ i = 1, ..., n.$$

$\sum_i P_i = 0$ $[\theta_i(t) \rightarrow \theta_i(t) + \Omega t]$.

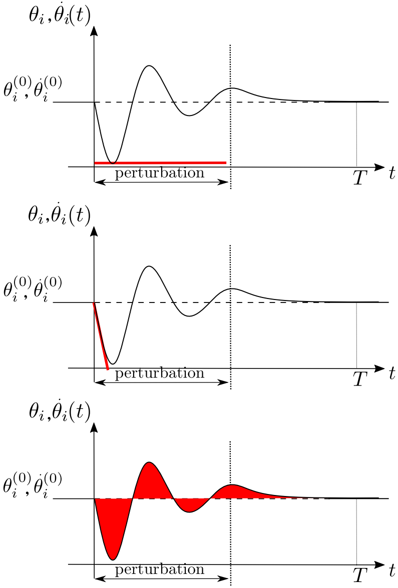
Perturbations $P_i \rightarrow P_i^{(0)} + \delta P_i(t)$.

Coupled Dynamical Systems: Example

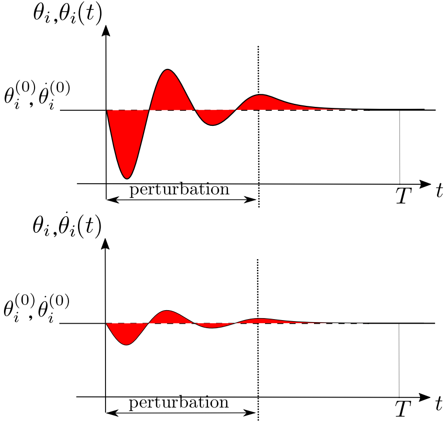


Quantifying Robustness

- Maximum of the response, $\max_t(\theta_i)$.
- Rate of change of frequency (RoCoF), $\ddot{\theta}_i$.
- Performance measure (quadratic integrals over the transient).



Performance measures



$$\mathcal{P}_1(T) = \sum_i \int_0^T |\theta_i(t) - \theta_i^{(0)}|^2 \mathrm{d}t \,,$$

$$\mathcal{P}_2(T) = \sum_i \int_0^T |\dot\theta_i(t) - \dot\theta_i^{(0)}|^2 \mathrm{d}t \,.$$

$$\mathcal{P}_{1,2}^{\infty} = \mathcal{P}_{1,2}(T \to \infty)\,.$$

Noisy disturbances → divide by T.

Perturbations : $P_i \to P_i^{(0)} + \delta P_i(t)$.

Response to Perturbations: Linearization

Linear response Perturbation of the natural frequencies.

- $P_i(t) = P_i^{(0)} + \delta P_i(t) \rightarrow \theta_i(t) = \theta_i^{(0)} + \delta\theta_i(t)$:

$$\boldsymbol{M}\delta\ddot{\boldsymbol{\theta}}(t) + \boldsymbol{D}\delta\dot{\boldsymbol{\theta}}(t) = \delta\boldsymbol{P}(t) - \mathbb{L}(\{\theta_i^{(0)}\})\delta\boldsymbol{\theta}(t) \ ,$$

$\mathbb{L}(\{\theta_i^{(0)}\})$: the weighted Laplacian matrix,

$$\mathbb{L}_{ij}(\{\theta_i^{(0)}\}) = \begin{cases} -a_{ij}\cos(\theta_i^{(0)} - \theta_j^{(0)}) \,, & i \neq j \,, \\ \sum_k a_{ik}\cos(\theta_i^{(0)} - \theta_k^{(0)}) \,, & i = j \,. \end{cases}$$

Topology $\rightarrow a_{ij}$.
Steady state $\rightarrow \{\theta_i^{(0)}\}$.

Response to Perturbations: Time Scales

Intrinsic Time Scales

- Network relaxation: $1/\lambda_\alpha$ with $\{\lambda_\alpha\}$ the eigenvalues of $\mathbb{L}$.
- Individual elements: m/d.

Perturbation Time Scale

- Correlation time of the external perturbation $\delta\boldsymbol{P}(t)$.

Noisy time correlated perturbations

- $\overline{\delta P_i(t)P_j(t')} = \delta P_{0i}^2\delta_{ij}\exp[-|t-t'|/\tau_0]$.

Correlation time $\rightarrow \tau_0$.

Response to Perturbations: Specific δP_0

Noisy time correlated perturbations

$$\overline{\mathcal{P}_1} \;=\; \sum_\alpha \frac{\sum_{i\in N_\mathrm{n}} \delta P_{0i}^2\, u_{\alpha,i}^2(\tau_0 + m/d)}{\lambda_\alpha(\lambda_\alpha \tau_0 + d + m/\tau_0)}\,.$$

N_n: noisy nodes.

Averaged perturbations

$$\langle \delta P_{0,i} \delta P_{0,j}\rangle = \delta_{ij}\langle \delta P_0^2\rangle$$

Short and long correlation time

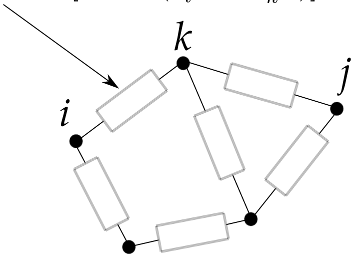
$$\overline{\mathcal{P}_1^\infty} \;\simeq\; \begin{cases} \langle \delta P_0^2\rangle\, \tau_0 \sum_{\alpha\geq 2} \lambda_\alpha^{-1}/2\,, & \lambda_\alpha \tau_0\,, m/d \ll 1\,, \forall\alpha\,, \\ \langle \delta P_0^2\rangle \sum_{\alpha\geq 2} \lambda_\alpha^{-2}\,, & \lambda_\alpha \tau_0\,, m/d \gg 1\,, \forall\alpha\,. \end{cases}$$

Resistance Distance

$$\Omega_{ij}^{(1)} = \mathbb{L}_{ii}^{\dagger} + \mathbb{L}_{jj}^{\dagger} - \mathbb{L}_{ij}^{\dagger} - \mathbb{L}_{ji}^{\dagger} = \sum_{\alpha\geq 2} \frac{(u_{\alpha,i} - u_{\alpha,j})^2}{\lambda_\alpha} \,.$$

$\mathbb{L}^{\dagger}$: pseudo inverse of $\mathbb{L}$ (because of $\lambda_1 = 0$).

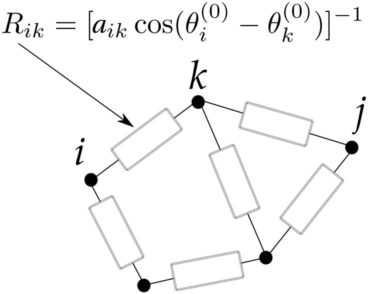
$$R_{ik} = [a_{ik}\cos(\theta_i^{(0)} - \theta_k^{(0)})]^{-1}$$



Klein and Randić, *J. Math. Chem.* **12**, 81 (1993).

Kirchhoff Index

$$Kf_1 = \sum_{i<j} \Omega_{ij}^{(1)} = n \sum_{\alpha \geq 2} \lambda_\alpha^{-1} \ .$$



Resistance Distances and Kf_p's

Generalized Resistance Distances

$$\Omega_{ij}^{(p)} = \mathbb{L}'^{\dagger}_{ii} + \mathbb{L}'^{\dagger}_{jj} - \mathbb{L}'^{\dagger}_{ij} - \mathbb{L}'^{\dagger}_{ji}$$
$$= \sum_{\alpha \geq 2} \frac{(u_{\alpha,i} - u_{\alpha,j})^2}{\lambda_\alpha^p} \,,$$
$$\mathbb{L}' = \mathbb{L}^p \,.$$

Generalized Kirchhoff Indices

$$Kf_p = \sum_{i<j} \Omega_{ij}^{(p)} = n \sum_{\alpha \geq 2} \lambda_\alpha^{-p} \,.$$

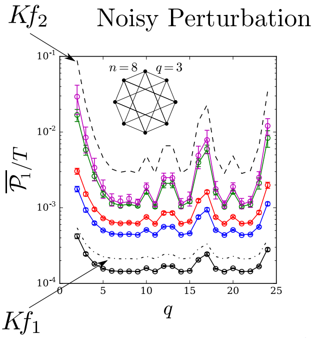
Response to Perturbations: Global Robustness

Averaged perturbations

$$\langle \delta P_{0,i} \delta P_{0,j} \rangle = \delta_{ij} \langle \delta P_0^2 \rangle$$

Noisy time correlated perturbations

$$\overline{\mathcal{P}_1^{\infty}} \simeq \begin{cases} \langle \delta P_0^2 \rangle \, \tau_0 \, Kf_1/n \,, & \lambda_\alpha \tau_0 \ll 1 \,, \forall \alpha \,, \\ \langle \delta P_0^2 \rangle \; Kf_2/n \,, & \lambda_\alpha \tau_0 \gg 1 \,, \forall \alpha \,. \end{cases}$$



Local perturbations

$$\delta P_{0,i} = \delta_{ik} \delta P_0$$

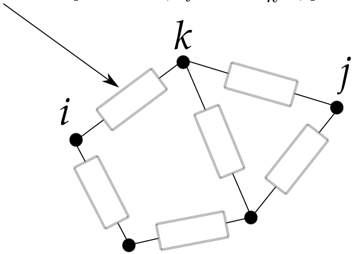
Noisy time correlated perturbations

$$\overline{\mathcal{P}_1^\infty}(k) \simeq \begin{cases} \delta P_0^2\, \tau_0 \sum_{\alpha\geq 2} \frac{u_{\alpha,k}^2}{\lambda_\alpha}\,, & \lambda_\alpha \tau_0 \ll 1\,, \forall\alpha\,, \\ \delta P_0^2 \sum_{\alpha\geq 2} \frac{u_{\alpha,k}^2}{\lambda_\alpha^2}\,, & \lambda_\alpha \tau_0 \gg 1\,, \forall\alpha\,. \end{cases}$$

Centralities

$$C_p(k) = \left[n^{-1} \sum_j \Omega_{kj}^{(p)} \right]^{-1} = \left[\sum_{\alpha \geq 2} \frac{u_{\alpha,k}^2}{\lambda_\alpha^p} + n^{-2} Kf_p \right]^{-1} .$$

$$R_{ik} = [a_{ik} \cos(\theta_i^{(0)} - \theta_k^{(0)})]^{-1}$$



Response to Perturbations: Local Vulnerabilities

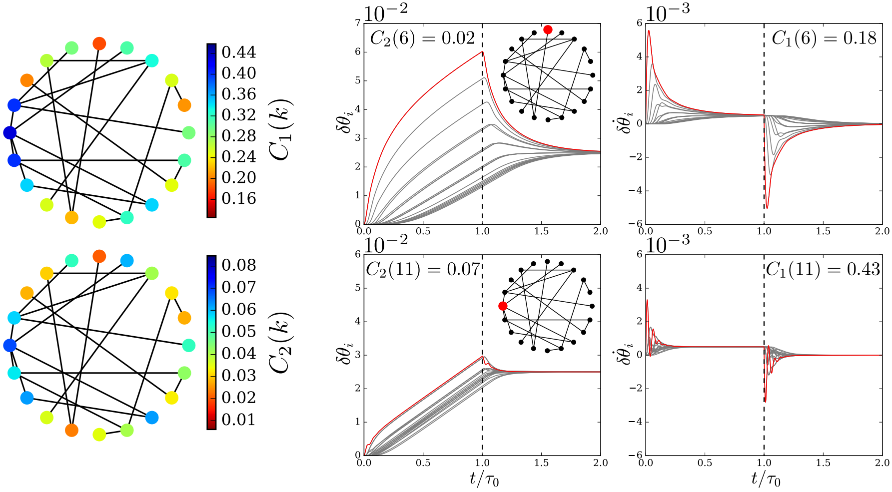
Local perturbations

$$\delta P_{0,i} = \delta_{ik}\delta P_0$$

Noisy time correlated perturbations

$$\overline{\mathcal{P}_1^{\infty}}(k) \simeq \begin{cases} \delta P_0^2\,\tau_0\,[C_1^{-1}(k) - n^{-2}Kf_1]\,, & \lambda_\alpha\tau_0 \ll 1\,,\forall\alpha\,, \\ \delta P_0^2\,[C_2^{-1}(k) - n^{-2}Kf_2]\,, & \lambda_\alpha\tau_0 \gg 1\,,\forall\alpha\,. \end{cases}$$

Specific Local Vulnerabilities and C_p's: Numerics



Physical Realization : European Electrical Grid

$\tau_0 \ll d/\lambda_\alpha \,, m/d$

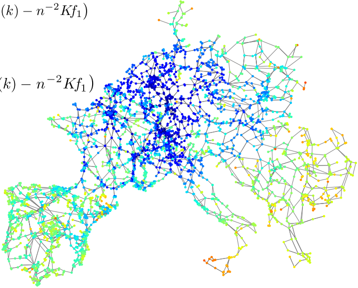
$\mathcal{P}_1^\infty \simeq \frac{\delta P_0^2 \tau_0}{d} \left(C_1^{-1}(k) - n^{-2} K f_1 \right)$

$\mathcal{P}_2^\infty \simeq \frac{\delta P_0^2 \tau_0}{dm} \frac{(n-1)}{n}$

$\tau_0 \gg d/\lambda_\alpha \;\; , m/d$

$\mathcal{P}_2^\infty \simeq \frac{\delta P_0^2}{d \tau_0} \left(C_1^{-1}(k) - n^{-2} K f_1 \right)$

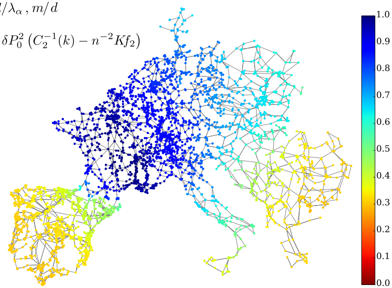
$C_1(i)/\max[C_1(i)]$



$C_2(i)/\max[C_2(i)]$

$\tau_0 \gg d/\lambda_\alpha \,, m/d$

$\mathcal{P}_1^\infty \simeq \delta P_0^2 \left(C_2^{-1}(k) - n^{-2} K f_2 \right)$



Transient Response – Conclusion

Global Robustness

- Generalized Kirchhoff Indices, Kf_p's.

Local Vulnerabilities

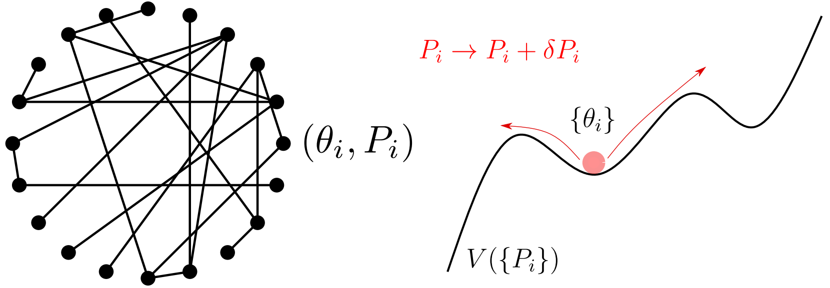
- Generalized Resistance Centralities, C_p's.
- Establish a ranking of the nodes.

$\rightarrow$ p depends on which performance measures you are interested in and on the correlation time of the perturbation.

Inertia

- No effect on performance measures in both asymptotics in τ_0 except for frequencies and short τ_0.

Introduction – Transitions



Transitions – Steady State Identification

Second-order Kuramoto

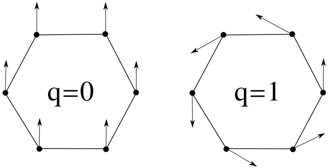
$$m_i\ddot{\theta}_i + d_i\dot{\theta}_i = P_i - \sum_j a_{ij}\sin(\theta_i - \theta_j) \ \ , \ i = 1, ..., n.$$

$a_{ij} = a_{ji} \geq 0$.

Steady-state solutions Synchronous state $\{\theta_i^{(0)}\}$ such that:

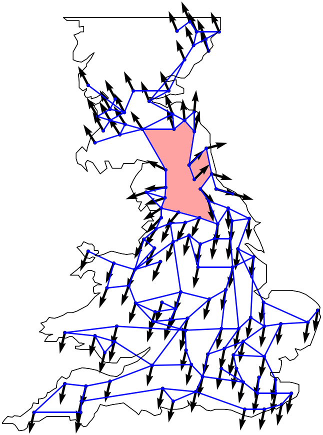
$$P_i = \sum_j a_{ij}\sin(\theta_i^{(0)} - \theta_j^{(0)}) \ \ , \ i = 1, ..., n.$$

Winding number $q = (2\pi)^{-1}\sum_{i\in c}|\theta_{i+1} - \theta_i|_{[-\pi,\pi]}$.



Delabays, Coletta, Jacquod, Journal of Mathematical Physics **58**, 032703 (2017).

Meshed network

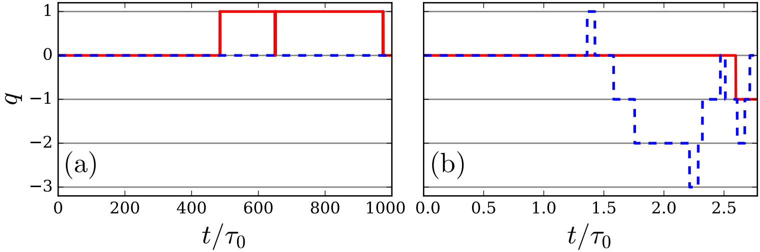


Delabays, MT, Jacquod, Chaos **27**, 103109 (2017).

Transitions – Noisy Environment

Noisy natural frequencies

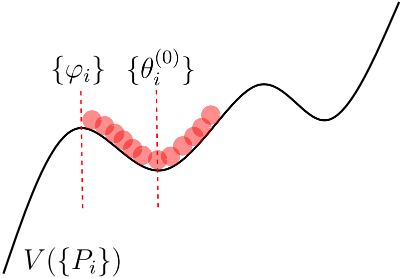
- $\langle \delta P_i(t) \rangle = 0$,
- $\langle \delta P_i(t) \delta P_j(t') \rangle = \delta_{ij}\, \delta P_0^2\, \exp[-|t-t'|/\tau_0]$.



Transitions – Escape Criterion

Noisy natural frequencies

- $\langle \delta P_i(t) \rangle = 0$,
- $\langle \delta P_i(t) \delta P_j(t') \rangle = \delta_{ij}\, \delta P_0^2 \, \exp[-|t-t'|/\tau_0]$.



Escape criterion $\langle \delta \boldsymbol{\theta}^2 \rangle \cong \|\boldsymbol{\theta}^{(0)} - \boldsymbol{\varphi}\|_2^2$

Transitions – Linear Response

Escape criterion $\langle \delta\theta^2 \rangle \cong \|\boldsymbol{\theta}^{(0)} - \boldsymbol{\varphi}\|_2^2$

Noisy natural frequencies

- $\langle \delta P_i(t) \rangle = 0$,
- $\langle \delta P_i(t) \delta P_j(t') \rangle = \delta_{ij}\, \delta P_0^2 \, \exp[-|t - t'|/\tau_0]$.

$$\lim_{t\to\infty} \langle \boldsymbol{\delta\theta}^2(t) \rangle = \delta P_0^2 \sum_{\alpha \geq 2} \frac{\tau_0 + m/d}{\lambda_\alpha(\lambda_\alpha \tau_0 + d + m/\tau_0)} \,.$$

Transitions – Distance to Closest Saddle

Escape criterion $\langle \delta\boldsymbol{\theta}^2 \rangle \cong \|\boldsymbol{\theta}^{(0)} - \boldsymbol{\varphi}\|_2^2$

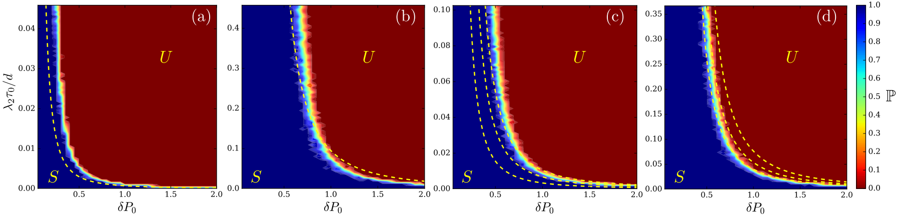
Single cycle
For identical natural frequencies:

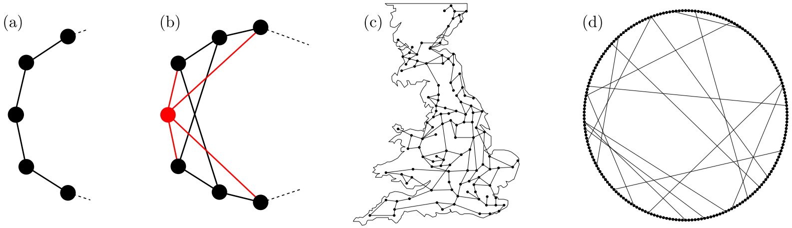
$$\Delta^2 = \left\|\boldsymbol{\theta}^{(0)} - \boldsymbol{\varphi}\right\|_2^2 = \frac{n(n^2-1)}{12(n-2)^2}\pi^2 \,.$$

Complex networks
Find saddles numerically, e.g. Newton-Raphson.

Delabays, MT, Jacquod, Chaos **27**, 103109 (2017).

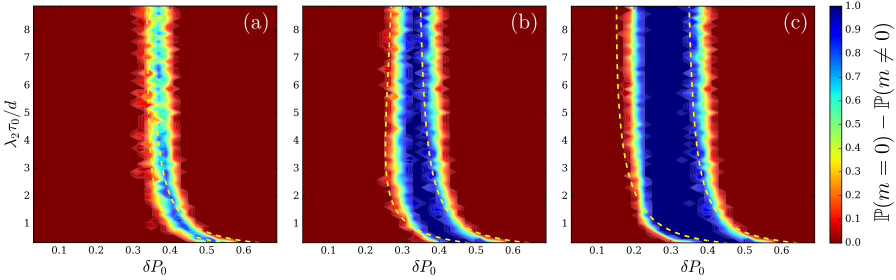
Transitions – Prediction





Transitions – Prediction

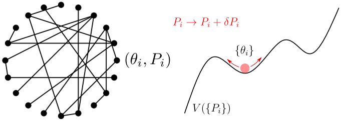
Effect of inertia



Overall Conclusion

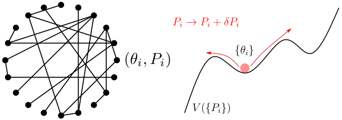
Small-signal response

- Response of coupled oscillators to external perturbations.
- Global network robustness $\rightarrow Kf_p$.
- Local network vulnerabilities $\rightarrow C_p,\ Kf_p$.



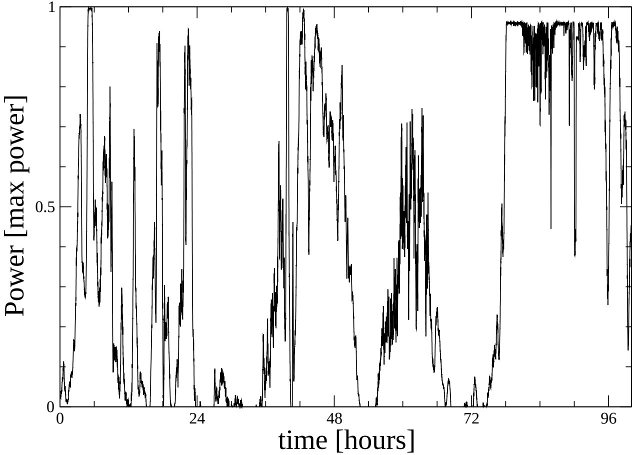
Transitions

- Estimate for escape probability based on distances between stable and unstable fixed points.
- Estimate for first escape time.



Non-Gaussian noise propagation

Time-correlated fluctuations of wind turbines production



Non-Gaussian noise propagation

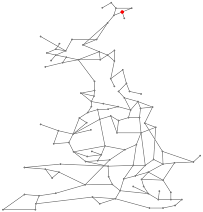
Time-correlated non-Gaussian noise

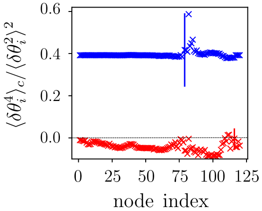
$$m_i\ddot{\theta}_i + d_i\dot{\theta}_i = P_i - \sum_j a_{ij}\sin(\theta_i - \theta_j) \ \ , \ i = 1, ..., n.$$

Non-Gaussian noise propagation

Time-correlated non-Gaussian noise

$$\lim_{\tau_0 \to 0} \qquad \langle \delta\theta_i^p \rangle = \left(\sigma \sum_\alpha \frac{u_{\alpha,i_0} u_{\alpha,i}}{\lambda_\alpha} \right)^p$$



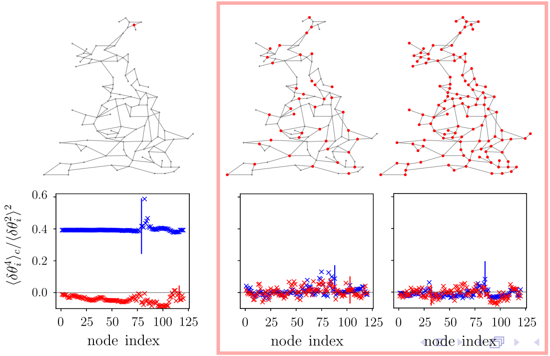


Non-Gaussian noise propagation

Time-correlated non-Gaussian noise

$$\lim_{\tau_0\to 0} \quad \langle\delta\theta_i^4\rangle = \sum_{i_0=1}^{M}\left(\sigma\sum_\alpha \frac{u_{\alpha,i_0}u_{\alpha,i}}{\lambda_\alpha}\right)^4$$

$$+3\sum_{i_0<j_0}\left(\sigma\sum_\alpha \frac{u_{\alpha,i_0}u_{\alpha,i}}{\lambda_\alpha}\right)^2\left(\sigma\sum_\beta \frac{u_{\beta,j_0}u_{\beta,i}}{\lambda_\beta}\right)^2, \qquad (5)$$



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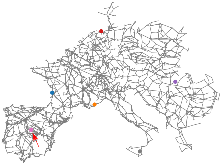
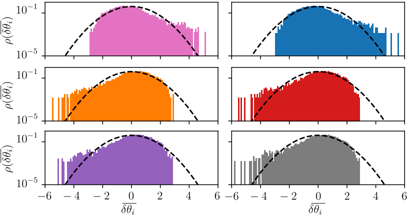
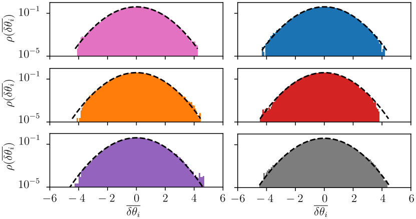


single source

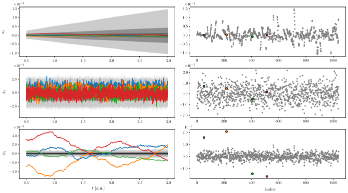
multiple sources



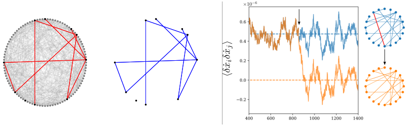


Hot topics

- Line fault identification
  - Delabays, Pagnier, MT New Journal of Physics **23**(4), 043037 (2021)
  - Delabays, Pagnier, MT arXiv:2202.08317 (2022)



- Network inference from partial observation
  - MT, Delabays, Jacquod Chaos **31**(10), 103117 (2021)
  - MT, Delabays Journal of Physics: Complexity **2**(2), 025016 (2021)
  - Delabays, MT IFAC-PapersOnLine **54**(9), 696-700 (2021)



- Forced oscillations → ongoing work...

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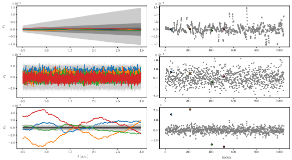

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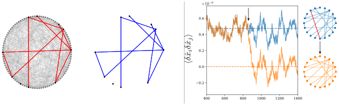

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→ BLABS talk April 11.