Periodic coupling inhibits second-order consensus on networks

Melvyn Tyloo

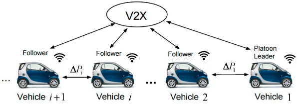
Networks 2021: A Joint Sunbelt and NetSci Conference.

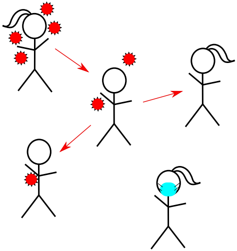
website: melvyntyloo.com

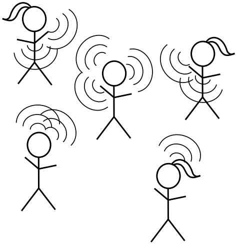
melvyn.tyloo@gmail.com — July 5, 2021

F. Baumann, I.M. Sokolov, MT, Physical Review E **102** (5), 052313 (2020)

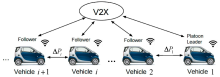
Complex Networks

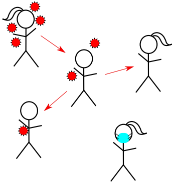


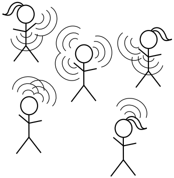




Time Dependent Networks







Time dependent networks
Diffusion-like processes

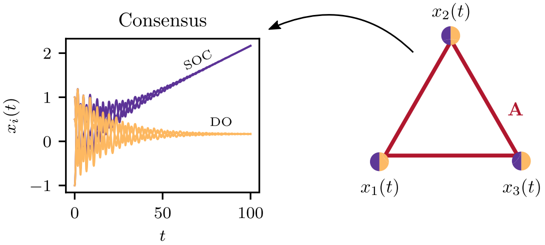
A. Li, S. P. Cornelius, Y.-Y. Liu, L. Wang, A.-L. Barabási, Science **358**(6366), 1042-1046 (2017).

Damped Oscillators (DO)

$$\ddot{x}_i + d\,\dot{x}_i = -\sum_j A_{ij}(x_i - x_j) \quad i = 1, \dots n\,. \tag{1}$$

Second Order Consensus (SOC)

$$\ddot{x}_i = -\sum_j A_{ij}[\gamma\,(x_i - x_j) + \mu\,(\dot{x}_i - \dot{x}_j)] \quad i = 1, \dots n\,. \tag{2}$$



Consensus on time-dependent networks

Damped Oscillators (DO)

$$\ddot{x}_i + d\,\dot{x}_i = -\sum_j W_{ij}(t)(x_i - x_j) \quad i = 1, \ldots n\,. \tag{3}$$

Second Order Consensus (SOC)

$$\ddot{x}_i = -\sum_j W_{ij}(t)[\gamma\,(x_i - x_j) + \mu\,(\dot{x}_i - \dot{x}_j)] \quad i = 1, \ldots n\,. \tag{4}$$

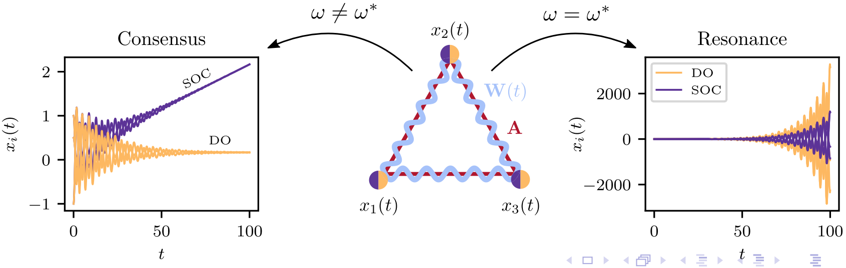
Simplest time-dependence $\rightarrow$ $W_{ij} = f(t)A_{ij}$

Damped Oscillators (DO)

$$\ddot{x}_i + d\,\dot{x}_i = -\sum_j W_{ij}(t)(x_i - x_j) \quad i = 1, ...n\,. \tag{5}$$

Second Order Consensus (SOC)

$$\ddot{x}_i = -\sum_j W_{ij}(t)[\gamma\,(x_i - x_j) + \mu\,(\dot{x}_i - \dot{x}_j)] \quad i = 1, ...n\,. \tag{6}$$



General Form

$$\ddot{\mathrm{x}} + d(t)\dot{\mathrm{x}} = -\mathbb{L}(t)(\gamma \mathrm{x} + \mu \dot{\mathrm{x}}) \,, \tag{7}$$

- DO: $d(t) = d \,,\ \mu = 0 \,.$
- SOC: $d(t) = 0 \,,\ \gamma,\ \mu \neq 0 \,.$

Time-dependence $\rightarrow \mathbb{L}(t) = f(t)\mathbb{L}^{(0)}$

$$\mathbb{L}^{(0)}_{ij} = \begin{cases} -A_{ij} \,, & i \neq j \,, \\ \sum_k A_{ik} \,, & i = j \,. \end{cases} \tag{8}$$

Eigenvalues $\rightarrow \lambda_\alpha(t) = f(t)\lambda_\alpha^{(0)}$

Consensus on time-dependent networks

Expansion over network modes $x_i = \sum_\alpha c_\alpha u_{\alpha,i}$

$$\rightarrow \ddot{c}_\alpha + k(t)\dot{c}_\alpha + \gamma\lambda_\alpha(t)c_\alpha = 0\,, \tag{9}$$

$k(t) = d(t) + \mu\lambda_\alpha(t)\,.$

Consensus on time-dependent networks

Expansion over network modes $x_i = \sum_\alpha c_\alpha u_{\alpha,i}$

$$\rightarrow \ddot{c}_\alpha + k(t)\dot{c}_\alpha + \gamma\lambda_\alpha(t)c_\alpha = 0\,, \tag{10}$$

$k(t) = d(t) + \mu\lambda_\alpha(t)\,.$

Parametric oscillator along each network mode.

$$\ddot{x} + d(t)\dot{x} + \omega^2(t)x = 0\,, \tag{11}$$

$\rightarrow$ tuning damping/natural frequency induces resonances.

L. D. Landau and E. M. Lifschitz, Lehrbuch der TheoretischenPhysik, Band I, Mechanik, 11th ed. (Akademie, Berlin, 1984),Vol. 1.

Theory – conditions for resonances

Coupling Network $\rightarrow \mathbb{L}(t) = f(t)\mathbb{L}^{(0)}$.

Time-dependence $f(t) = [1 + h\sin(\omega t)]$.

$$\rightarrow \ddot{c}_\alpha + k(t)\dot{c}_\alpha + \gamma\lambda_\alpha(t)c_\alpha = 0\,, \tag{12}$$

$$c_\alpha(t) = e^{-K(t)}q_\alpha(t)\,, \tag{13}$$

$$K(t) = \frac{1}{2}\int_0^t k(t')\mathrm{d}t'\,. \tag{14}$$

Substituting those into Eq. (12) yields

$$\ddot{q}_\alpha + \Omega_\alpha^2(t)q_\alpha = 0\,, \text{(Hill equation)} \tag{15}$$

$$\Omega_\alpha^2(t) = \gamma\lambda_\alpha(t) - k^2(t)/4 - \dot{k}(t)/2\,. \tag{16}$$

Theory – conditions for resonances

Coupling Network $\rightarrow \mathbb{L}(t) = f(t)\mathbb{L}^{(0)}$.

Time-dependence $f(t) = [1 + h\sin(\omega t)]$.

$$\ddot{q}_\alpha + \Omega_\alpha^2(t) q_\alpha = 0 \,, \text{(Hill equation)} \tag{17}$$

- Damped oscillator

$$\omega_\alpha^* = 2\omega_\alpha = 2\sqrt{\gamma\lambda_\alpha^{(0)} - \left(\frac{d}{2}\right)^2}\,, \tag{18}$$

- Second order consensus

$$\omega_\alpha^* = 2\omega_\alpha = 2\sqrt{\gamma\lambda_\alpha^{(0)} - \left(\frac{\mu\lambda_\alpha^{(0)}}{2}\right)^2}\,, \tag{19}$$

Theory – conditions for resonances

Coupling Network $\rightarrow \mathbb{L}(t) = f(t)\mathbb{L}^{(0)}\,.$

Time-dependence $f(t) = [1 + h\sin(\omega t)]\,.$

- Damped oscillator

$$\omega_\alpha^* = 2\omega_\alpha = 2\sqrt{\gamma\lambda_\alpha^{(0)} - \left(\frac{d}{2}\right)^2}\,, \tag{20}$$

- Second order consensus

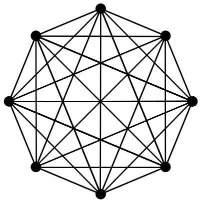
$$\omega_\alpha^* = 2\omega_\alpha = 2\sqrt{\gamma\lambda_\alpha^{(0)} - \left(\frac{\mu\lambda_\alpha^{(0)}}{2}\right)^2}\,, \tag{21}$$

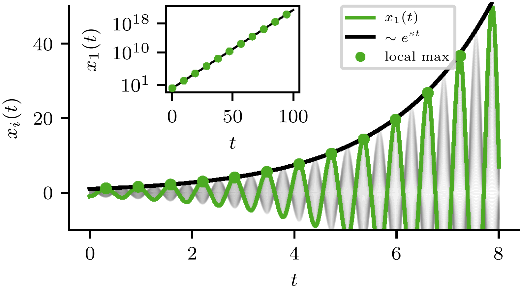
$$q_\alpha(t) = \quad a_\alpha(t)\cos[(\omega_\alpha + \varepsilon)t] + b_\alpha(t)\sin[(\omega_\alpha + \varepsilon)t]\,, \tag{22}$$

$\rightarrow$ conditions for resonance on h, ω_α, d, ϵ.

Numerics – conditions for resonances

Complete network (aka all-to-all, $n = 100$)





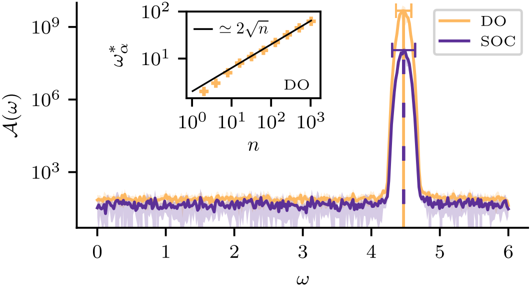
- DO:

$$\omega^* = 2\sqrt{\gamma n - \left(\frac{d}{2}\right)^2}, \tag{23}$$

- SOC:

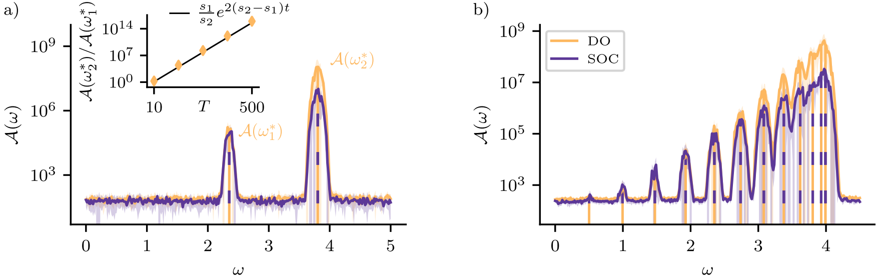
$$\omega^* = 2\sqrt{\gamma n - \left(\frac{\mu n}{2}\right)^2}. \tag{24}$$

Complete network (aka all-to-all, $n = 5$)



$$\mathcal{A}(\omega) = \left\langle \int_0^T \mathrm{x}^2(t)\,\mathrm{d}t \right\rangle_{\{\mathrm{IC}\}} . \tag{25}$$

Cycle networks ($n = 5$ and $n = 25$)



- DO:

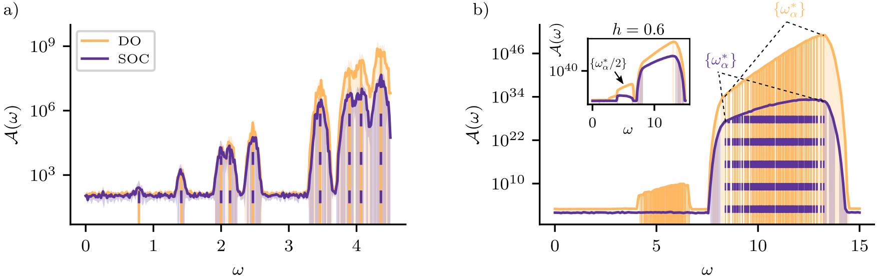
$$\omega_\alpha^* = 2\sqrt{2 - 2\cos(k_\alpha) - \left(\frac{d}{2}\right)^2} \tag{26}$$

- SOC:

$$\omega_\alpha^* = 2\sqrt{2 - \mu^2 + (2\mu - 2)\cos(k_\alpha) - \mu^2\cos^2(k_\alpha)} \tag{27}$$

Numerics – conditions for resonances

Erdős-Rényi networks ($n = 25$ and $n = 100$)



- DO:

$$\omega_\alpha^* = 2\omega_\alpha = 2\sqrt{\gamma\lambda_\alpha^{(0)} - \left(\frac{d}{2}\right)^2}\,, \tag{28}$$

- SOC:

$$\omega_\alpha^* = 2\omega_\alpha = 2\sqrt{\gamma\lambda_\alpha^{(0)} - \left(\frac{\mu\lambda_\alpha^{(0)}}{2}\right)^2}\,, \tag{29}$$

Numerics – conditions for resonances

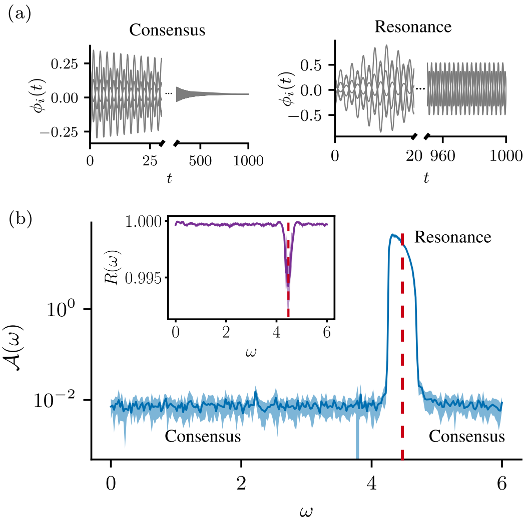
Nonlinear model: second order Kuramoto

$$\ddot{\phi}_i + d\,\dot{\phi}_i = \omega_i - \sum_{j=1}^{N} W_{ij}(t)\sin(\phi_i - \phi_j)\;, \tag{30}$$

$$\delta\ddot{\phi}_i + d\,\delta\dot{\phi}_i = -\sum_{j=1}^{N} W_{ij}(t)\cos(\phi_{i,0} - \phi_{j,0})(\delta\phi_{i,0} - \delta\phi_{j,0})\;. \tag{31}$$

Numerics – conditions for resonances

Kuramoto: complete graph ($n = 5$)



Conclusion

Consensus on time-dependent networks

- Time-dependent networks: average interaction not sufficient.
- Resonances for diffusion-like processes with second order.
- Devise algorithms to disrupt consensus.
- Inference algorithms.

Future work

- Consider local time-dependent coupling.

F. Baumann, I.M. Sokolov, MT, Physical Review E **102** (5), 052313 (2020)

Conclusion

Consensus on time-dependent networks

- Time-dependent networks: average interaction not sufficient.
- Resonances for diffusion-like processes with second order.
- Devise algorithms to disrupt consensus.
- Inference algorithms.

Future work

- Consider local time-dependent coupling.

Advertisement

- 10128. Reconstructing Network Structure from Partial Measurements
- 10823. Locating line and node disturbances in networks of diffusively coupled dynamical agents

F. Baumann, I.M. Sokolov, MT, Physical Review E **102** (5), 052313 (2020)