Coupled Oscillators vs. Opinion Formation

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Models for coupled oscillators, describing electric power grids or chemical reactions, and those for opinion formation among agents exchanging ideas on influence networks are mathematically similar. Even if questions arising in each topic differ, they might be answered using simlar tools.

Linear Oscillators

We consider a set of n oscillators, each with a continuous degree of freedom $x_i \in \mathbb{R}$, linearly coupled on a complex network whose dynamics is given by

$$\dot{x}_i = P_i(t) - \sum_j b_{ij}(x_i - x_j)\ , \quad i = 1, ...n\ .$$

P_i : natural frequencies.
b_{ij} : element of the adjacency of the coupling network.
In a vectorial form,

$$\dot{\mathbf{x}} = \mathbf{P} - \mathbb{L}\,\mathbf{x}\ .$$

$\mathbb{L}$: Laplacian of the network.
$\mathbf{x}^{(0)}$: Stable fixed point.
Remark : The Laplacian is weighted if one considers the linear response of a non-linear model, e.g. Kuramoto oscillators.

Questions

1. Which oscillator is the most vulnerable component of the system? [4, 5]
2. How to couple oscillators to increase robustness of the system? [3]

Quantifying Transient Excursions

We apply a perturbation in the natural frequency of a single oscillator as $\delta P_i(t) = \delta_{ik}\Theta(t)\Theta(\tau_0 - t)$, where τ_0 allows to visit different time scales of the system. To quantify fragility, we integrate the deviation from the initial fixed point over the whole transient response as,

$$\mathcal{P}_1 = \sum_i \int_0^\infty |x_i(t) - x_i^{(0)}|^2 \mathrm{d}t\ .$$

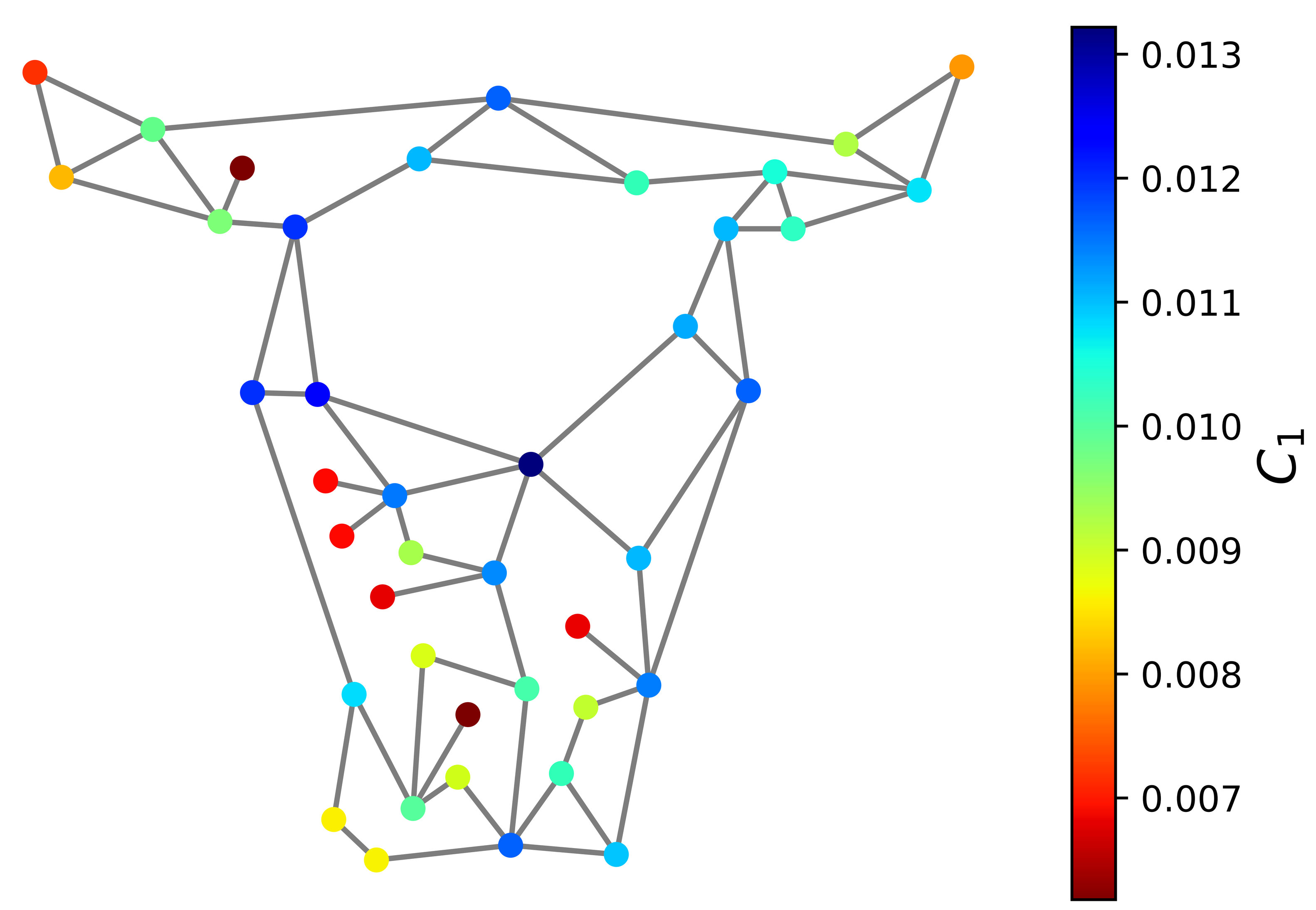
Generalized Resistance Distances A complex network metric can be defined as, [2]

$$\Omega_{ij}^{(p)} = \mathbb{L}'^{\dagger}_{ii} + \mathbb{L}'^{\dagger}_{jj} - \mathbb{L}'^{\dagger}_{ij} - \mathbb{L}'^{\dagger}_{ji}\ ,$$

with $\mathbb{L}' = \mathbb{L}^p$ where $\mathbb{L}^\dagger$ is the pseudo inverse of $\mathbb{L}$ (because of $\lambda_1 = 0$).
Generalized Resistance Centralities A closeness centrality follows as,

$$C_p(k) = \left[n^{-1} \sum_j \Omega_{kj}^{(p)} \right]^{-1}\ .$$

Vulnerabilities For short/long τ_0, performance measure $\mathcal{P}_1 \propto C_{1,2}^{-1}(k)$.



Taylor Model

We consider a set of n agents exchanging continuous opinions $x_i \in \mathbb{R}$ on an influence network. Some agents ($\notin V_s$) try to minimize their opinion differences with respect to their neighbors. Some other agents ($\in V_s$) have an additional bias that steers their opinion to an arbitrary value. The overall dynamics follows,

$$\dot{x}_i = -\sum_j b_{ij}(x_i - x_j)\ , \quad i \notin V_s\ ,$$
$$\dot{x}_i = -\sum_j b_{ij}(x_i - x_j) - \kappa[x_i - P_i(t)]\ , \quad i \in V_s\ .$$

V_s : set of stubborn agents. P_i : opinion biais for stubborn agents. κ : stubbornness.
In a vectorial form,

$$\dot{\mathbf{x}} = \kappa\,\mathbf{P} - \mathbb{L}^\kappa\,\mathbf{x}\ .$$

$\mathbb{L}^\kappa$: modified Laplacian.

Questions

1. Which agent can shift the consensus the most efficiently? [1]
2. How agents associate to each of two opposing stubborn agents? [1]
3. How two opposing stubborn agents influence opinion heterogeneity? [1]

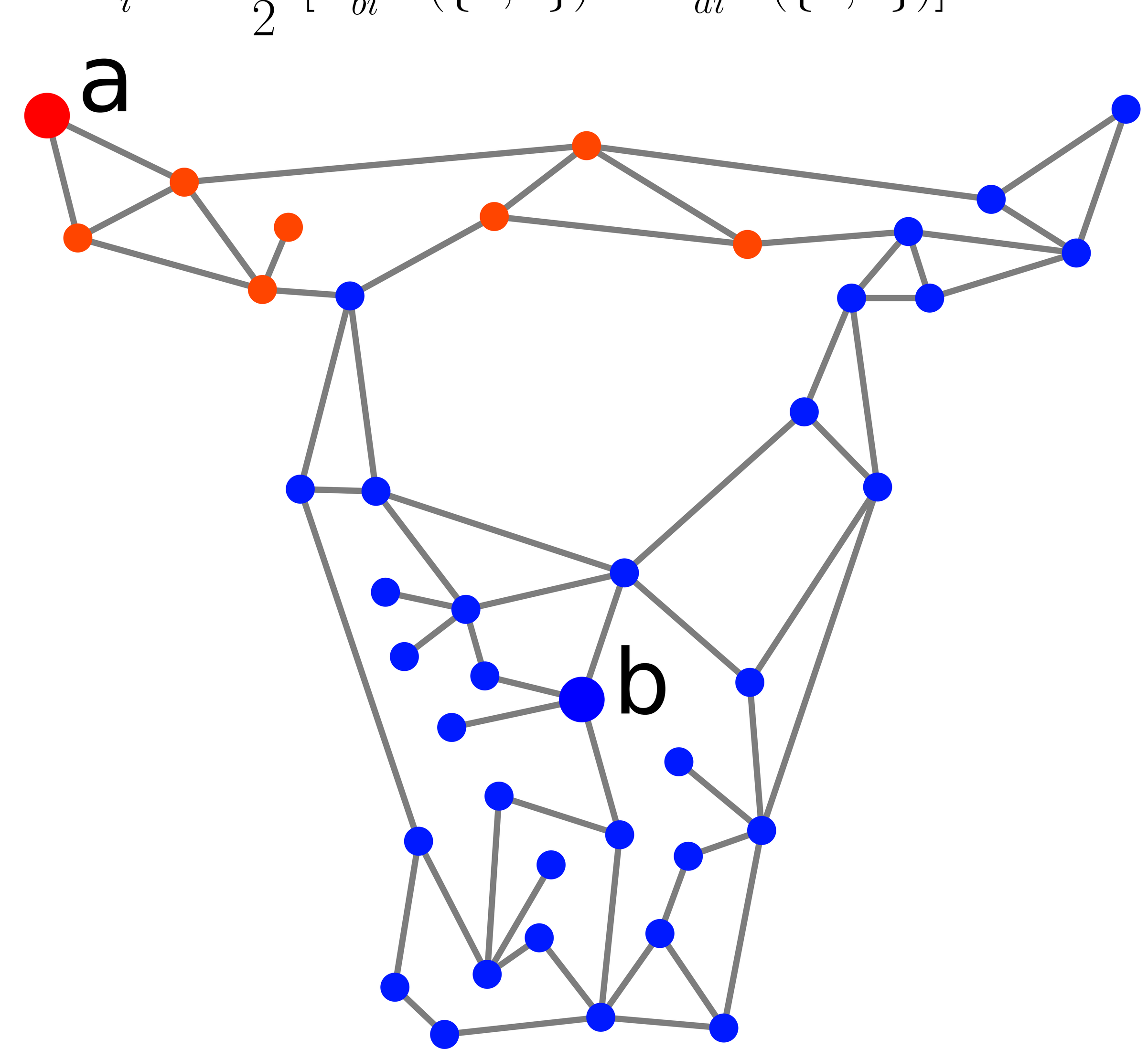
Opinion Association

We consider the case of two opposed stubborn agents i.e $P_a = -P_b = P$. Each agent is associated to a or b following the sign of their opinion in the final state $\mathbf{x}^\infty$. The association can be expressed analytically using modified resistance distances, [1]

$$\Omega_{ij}^{(\kappa,1)}(V_s) = [\mathbb{L}^\kappa]_{ii}^{-1} + [\mathbb{L}^\kappa]_{jj}^{-1} - [\mathbb{L}^\kappa]_{ij}^{-1} - [\mathbb{L}^\kappa]_{ji}^{-1}\ ,$$

where the index 1 denotes the first order modified resistance distance (MRD) as,

$$x_i^\infty = \frac{\kappa P}{2}[\Omega_{bi}^{(\kappa,1)}(\{a,b\}) - \Omega_{ai}^{(\kappa,1)}(\{a,b\})]\ .$$



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