Fluctuations in Complex Network-coupled Oscillators

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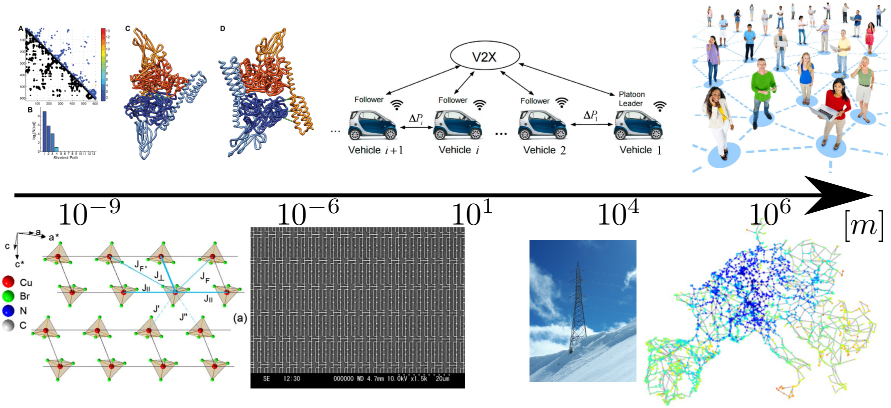
Introduction – Coupled Dynamical Systems



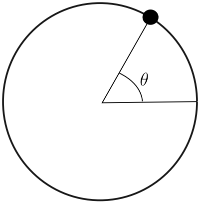
Sources: Phys. Rev. B **80**, 094411 (2009)
PLoS Comput Biol 11(6), e1004262 (2015)
Phys. Rev. Lett. **85**, 1974 (2000).
Transportation Research Part C: Emerging Technologies Volume 84, November 2017, Pages 21-47.
https://etranselec.ch/2019/10/07/Swissgri.html
https://researchoutreach.org/articles/opinion-dynamics-consensus-social-networks/

Synchronization

- Simple experiment: coupled metronomes
- Other examples:
  - Fireflies flashing in unison
  - People clapping their hands or walking on a bridge
  - Synchronization of the quantum phase of Josephson junctions arrays
  - Synchronization of the phase of the voltages in electric power grids
  - Consensus formation on influence networks
  - Vehicular platoon formation
  - Orchestra

Modelling Synchronization

Single phase oscillator: $\dot{\theta} = \omega$



Modelling Synchronization

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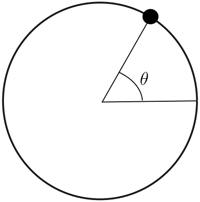
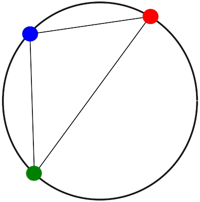


Coupled phase oscillators: $\dot{\theta}_i = \omega_i - \sum_j a_{ij} f(\theta_i - \theta_j)$



Modelling Synchronization

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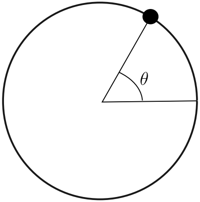

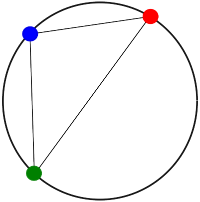

Coupled phase oscillators: $\dot{\theta}_i = \omega_i - \sum_j a_{ij} f(\theta_i - \theta_j)$

Synchronization: phase-locked $\dot{\theta}_i(t) = \dot{\theta}_j(t)\,,\ \forall i, j\,.$

Kuramoto model

$$\dot{\theta}_i = \omega_i - \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_i - \theta_j)\,, \text{for } i = 1, ..., N\,. \tag{1}$$

ω_i: natural frequencies.

Y. Kuramoto, Lecture Notes in Physics 39, International Symposium on Mathematical Problems in Theoretical Physics (1975).

Modelling of Synchronization

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ω_i: natural frequencies.

Order parameter

$$re^{i\,\psi} = N^{-1}\sum_{j=1}^{N} e^{i\,\theta_j} \tag{2}$$

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$$re^{i\,\psi} = N^{-1}\sum_{j=1}^{N} e^{i\,\theta_j} \tag{2}$$

ψ: average phase.

Illustration

$$\dot{\theta}_i = \omega_i - Kr\sin(\psi - \theta_i)\,,\text{for } i = 1, ..., N\,. \tag{3}$$

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Kuramoto model

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{N} a_{ij} \sin(\theta_i - \theta_j)\,, \text{for } i = 1, ..., N\,. \tag{4}$$

ω_i: natural frequencies.
a_{ij}: adjacency matrix.

Good survey Dörfler and Bullo, Automatica **50** (6), 1539-1564, (2014)

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$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{N} a_{ij} \sin(\theta_i - \theta_j)\,, \text{for } i = 1, ..., N\,. \tag{4}$$

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Multistability

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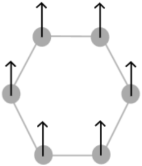

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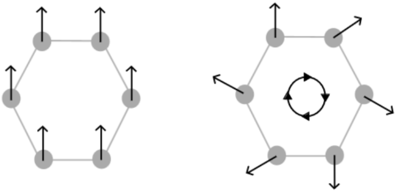

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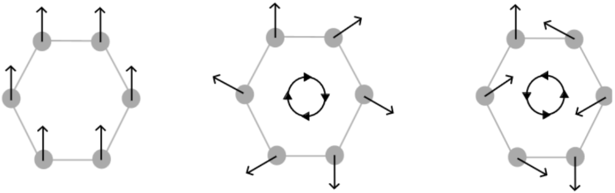

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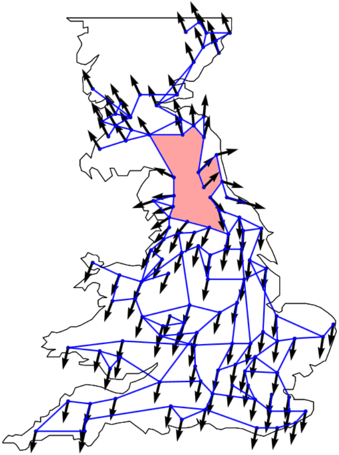
Synchronization on networks



Delabays, MT, Jacquod, Chaos **27**(10), 103109 (2017)

Dante R. Chialvo

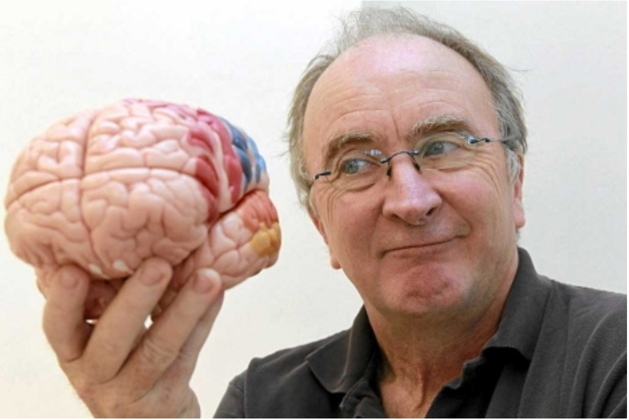


Figure: https://www.quantamagazine.org/toward-a-theory-of-self-organized-criticality-in-the-brain-20140403/

During a discussion between ABQ and SF: "...in the brain, nothing is in a synchronized state!"

Second order Kuramoto model

$$m_i\ddot{\theta}_i + d_i\dot{\theta}_i = P_i - \sum_j a_{ij}\sin(\theta_i - \theta_j) \ \ , \ i = 1, ..., n.$$

$a_{ij} = a_{ji} \geq 0$.
P_i : natural frequencies.
m_i : inertia.
d_i : damping.

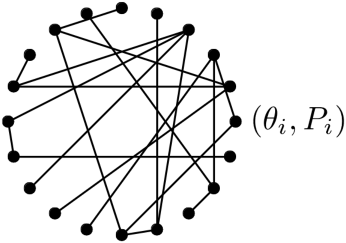
Electric Power Network (in the lossless line approximation)

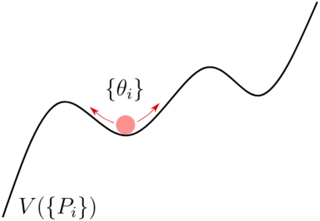
P_i : injected/consumed power.
$m_i = 0$: loads.
$m_i \neq 0$: generators.
$a_{ij}\sin(\theta_i - \theta_j)$: power flow from i to j .

J. A. Acebrón, L. L. Bonilla, Conrad J. Pérez Vicente, F. Ritort, and R. Spigler, Rev. Mod. Phys. **77**, 137 (2005)

Power system control and stability PM Anderson, AA Fouad - 1977

Robustness of synchronous networks





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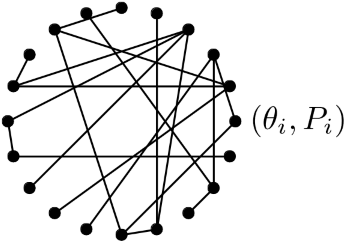
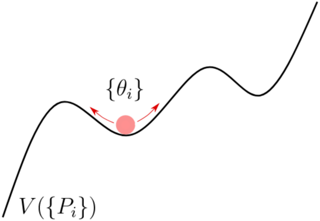
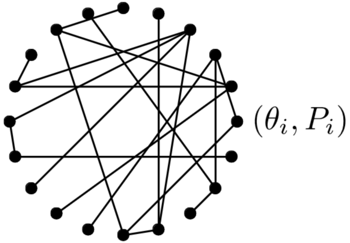
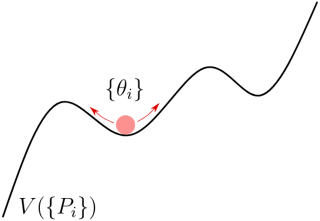




- Size of the basin of attraction

Robustness of synchronous networks





- Size of the basin of attraction
- Near equilibrium dynamics

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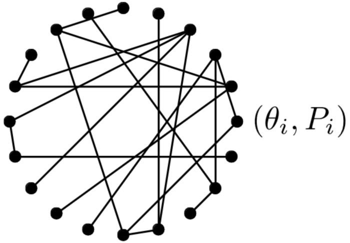

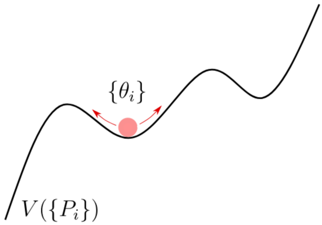

- Size of the basin of attraction
- Near equilibrium dynamics
- Transitions between fixed points

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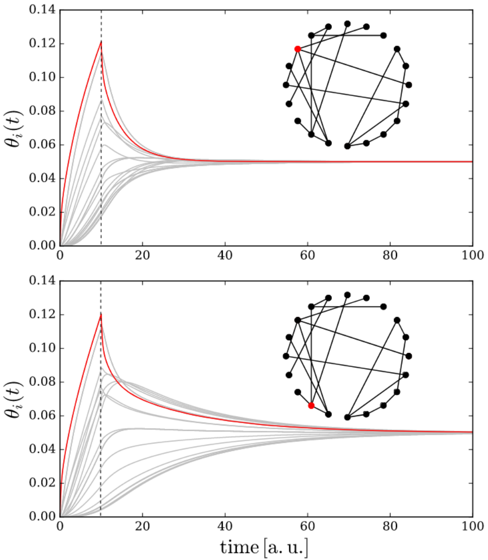

Robustness of synchronous networks to noise inputs

Near equilibrium dynamics

$$0 = \omega_i - \sum_{j=1}^{N} a_{ij} \sin(\theta_i^{(0)} - \theta_j^{(0)}) \,, \text{for } i = 1, ..., N \,. \tag{5}$$

Near equilibrium dynamics

$$0 = \omega_i - \sum_{j=1}^{N} a_{ij} \sin(\theta_i^{(0)} - \theta_j^{(0)}) \,, \text{for } i = 1, ..., N \,. \tag{5}$$

$$\delta\dot{\theta}_i = -\sum_{j=1}^{N} a_{ij} \cos(\theta_i^{(0)} - \theta_j^{(0)})(\delta\theta_i - \delta\theta_j) + \sum_m G_{im}\, \xi_m(t) \;. \tag{6}$$

$\xi_i(t)$: Noise inputs.

Near equilibrium dynamics

$$\boldsymbol{\delta\dot{\theta}} = -\mathbb{L}(\{\theta_k^{(0)}\})\boldsymbol{\delta\theta} + \mathbf{G}\,\boldsymbol{\xi}(t)\,, \text{for } i = 1, ..., N\,. \tag{7}$$

$\mathbb{L}(\{\theta_k^{(0)}\})$: Jacobian, with eigenvalues $\lambda_1 = 0 < \lambda_2 \leq ... \leq \lambda_N$.

Near equilibrium dynamics

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Solution

$$\delta\theta_i(t) = \sum_\alpha e^{-\lambda_\alpha\, t} \int_0^t e^{\lambda_\alpha\, t'} \sum_{j,m} G_{mj}\, \xi_j(t') u_{\alpha,m} \mathrm{d}t'\, u_{\alpha,i}\,. \tag{8}$$

Robustness of synchronous networks to noise inputs

Uncorrelated white noise

$$\langle \xi_i(t)\xi_j(t')\rangle = \xi_0^2\,\tau_0\,\delta_{ij}\,\delta(t-t') \quad (9)$$

$$G_{ij} = \delta_{ij} \quad (10)$$

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$$G_{ij} = \delta_{ij} \tag{10}$$

Variance at node *i*

$$\langle \delta\theta_i^2\rangle = \frac{\xi_0^2\,\tau_0}{2}\sum_\alpha \frac{u_{\alpha,i}^2}{\lambda_\alpha} \tag{11}$$

Robustness of synchronous networks to noise inputs

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Variance at node i

$$\langle \delta\theta_i^2\rangle = \frac{\xi_0^2\,\tau_0}{2}\sum_\alpha \frac{u_{\alpha,i}^2}{\lambda_\alpha} \quad (11)$$

Average variance in the network

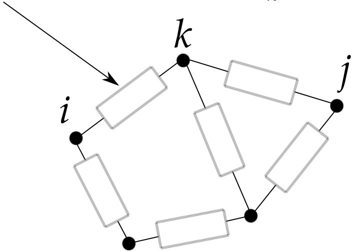
$$N^{-1}\sum_i \langle \delta\theta_i^2\rangle = \frac{\xi_0^2\,\tau_0}{2N}\sum_\alpha \frac{1}{\lambda_\alpha} \quad (12)$$

Resistance Distance

$$\Omega_{ij}^{(1)} = \mathbb{L}_{ii}^{\dagger} + \mathbb{L}_{jj}^{\dagger} - \mathbb{L}_{ij}^{\dagger} - \mathbb{L}_{ji}^{\dagger} = \sum_{\alpha \geq 2} \frac{(u_{\alpha,i} - u_{\alpha,j})^2}{\lambda_\alpha} \,.$$

$\mathbb{L}^{\dagger}$: pseudo inverse of $\mathbb{L}$ (because of $\lambda_1 = 0$).

$$R_{ik} = [a_{ik} \cos(\theta_i^{(0)} - \theta_k^{(0)})]^{-1}$$



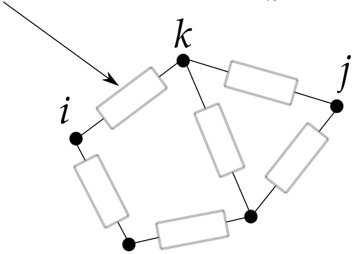
Klein and Randić, *J. Math. Chem.* **12**, 81 (1993).

Resistance Distance and Kf_1's

Kirchhoff Index

$$Kf_1 = \sum_{i<j} \Omega_{ij}^{(1)} = n \sum_{\alpha \geq 2} \lambda_\alpha^{-1} \ .$$

$$R_{ik} = [a_{ik} \cos(\theta_i^{(0)} - \theta_k^{(0)})]^{-1}$$



Resistance Distances and Kf_p's

Generalized Resistance Distances

$$\begin{aligned}\Omega_{ij}^{(p)} &= \mathbb{L}'^{\dagger}_{ii} + \mathbb{L}'^{\dagger}_{jj} - \mathbb{L}'^{\dagger}_{ij} - \mathbb{L}'^{\dagger}_{ji} \\ &= \sum_{\alpha \geq 2} \frac{(u_{\alpha,i} - u_{\alpha,j})^2}{\lambda_\alpha^p}\,, \\ \mathbb{L}' &= \mathbb{L}^p\,.\end{aligned}$$

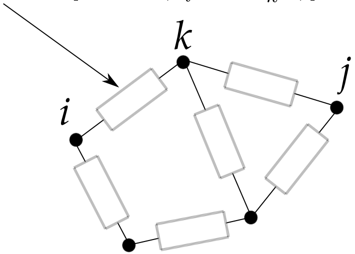
Generalized Kirchhoff Indices

$$Kf_p = \sum_{i<j} \Omega_{ij}^{(p)} = n \sum_{\alpha \geq 2} \lambda_\alpha^{-p}\,.$$

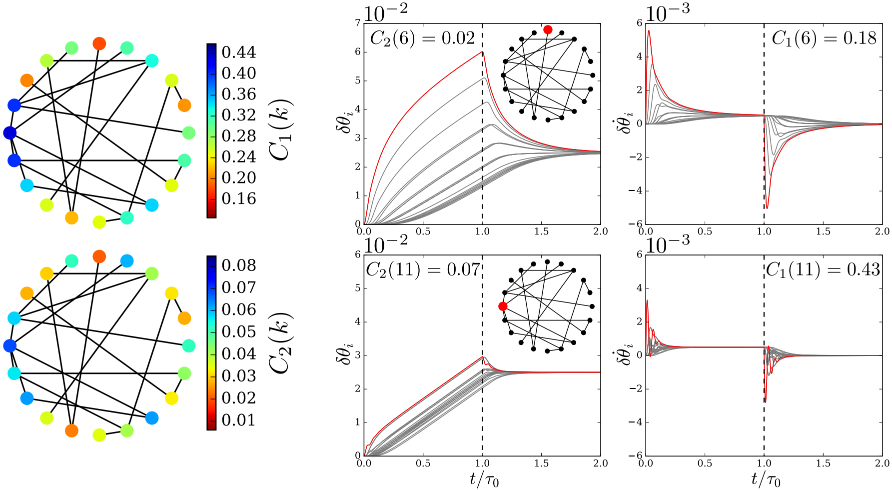
Centralities

$$C_p(k) = \left[n^{-1} \sum_j \Omega_{kj}^{(p)} \right]^{-1} = \left[\sum_{\alpha \geq 2} \frac{u_{\alpha,k}^2}{\lambda_\alpha^p} + n^{-2} Kf_p \right]^{-1} .$$

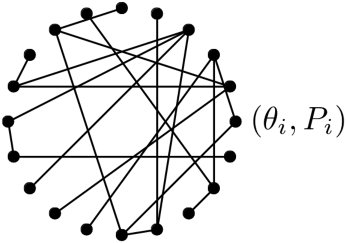
$$R_{ik} = [a_{ik} \cos(\theta_i^{(0)} - \theta_k^{(0)})]^{-1}$$

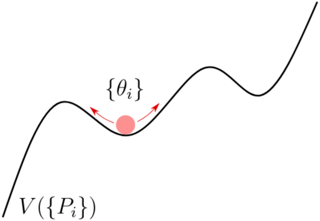


Local Vulnerabilities and C_p's: Numerics



Robustness of synchronous networks





- Size of the basin of attraction
- Near equilibrium dynamics
- Transitions between fixed points

Large fluctuations

Indicator Mode Approximation

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{N} A_{ij} \sin(\theta_i - \theta_j) + \sum_m G_{im}\, \xi_m(t)\,, \text{for } i = 1, ..., N\,. \qquad (13)$$

Hindes, Schwartz, MT, arxiv:2023.13434 (2023).

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Large fluctuations in the small noise limit $\rightarrow$ Numerically minimize the action (two-boundary problem).

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Large fluctuations in the small noise limit $\rightarrow$ Numerically minimize the action (two-boundary problem).
$\theta_i(t) = \theta_i^* + a(t)\Delta\theta_i \; \forall i$, where $\Delta\theta_i \equiv \theta_i^s - \theta_i^*$

$$\dot{a} = F(a) + \frac{\sum_i \Delta\theta_i \sum_m G_{im}\xi_m(t)}{\sum_l \Delta\theta_l^2}, \quad \text{where}$$

$$F(a) = \frac{\sum_i \Delta\theta_i \Big[\omega_i + K \sum_j A_{ij} \sin\big(\theta_j^* - \theta_i^* + a(\Delta\theta_j - \Delta\theta_i)\big)\Big]}{\sum_l \Delta\theta_l^2}. \qquad (14)$$

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Large fluctuations

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$$\langle \eta(t)\eta(t')\rangle = \sigma^2 \sum_m Q_m^2\, \delta(t-t'), \quad \text{where}$$
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Fokker-Planck equation for the probability flux of a,

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial a}\big[F(a)P\big] + \frac{1}{2}\frac{\partial^2}{\partial a^2}\big[\sigma^2 \sum_m Q_m^2 \, P\big]. \tag{16}$$

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WKB form: $P \sim \exp[-2S(a,t)/\sigma^2]$.

Large fluctuations

Indicator Mode Approximation

$\partial S/\partial t = 0$. In this case $H = 0$, and therefore

$$\frac{\partial S}{\partial a} = \frac{dS}{da} = -\frac{F(a)}{\sum_m Q_m^2}. \tag{17}$$

$$\begin{aligned} S &= \frac{\sum_l \Delta\theta_l^2}{\sum_{m,i}(G_{mi}\Delta\theta_i)^2}\, S_{\rm DB}, \quad \text{where} \\ S_{\rm DB} &= \sum_i \Delta\theta_i \Big[-\omega_i + K \sum_{\substack{j,\\ \Delta\theta_i \neq \Delta\theta_j}} \frac{A_{ij}}{\Delta\theta_j - \Delta\theta_i} \cdot \big(\cos(\theta_j^s - \theta_i^s) - \cos(\theta_j^* - \theta_i^*)\big)\Big]. \end{aligned} \tag{18}$$

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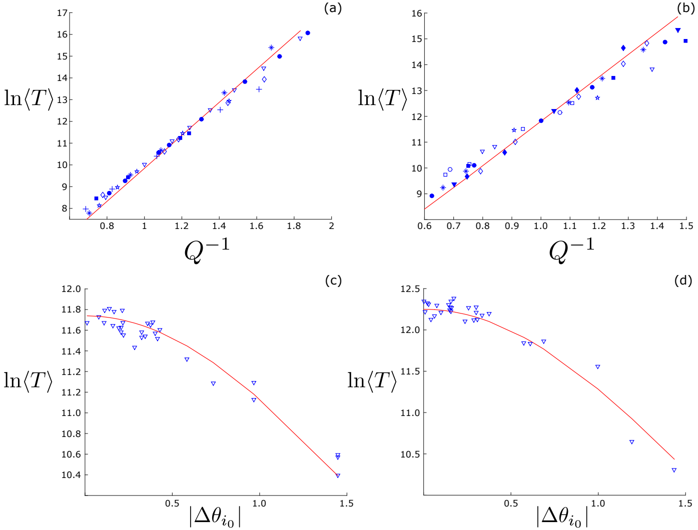
$$Q = \frac{\Delta\boldsymbol{\theta}^T G G^T \Delta\boldsymbol{\theta}}{\Delta\boldsymbol{\theta}^T \Delta\boldsymbol{\theta}}\,, \quad \ln\langle T\rangle \approx \frac{2S_{DB}}{\sigma^2 Q} + \text{constant}. \tag{19}$$

Indicator Mode Approximation



Conclusion and future work

So far

- Linearization around an equilibrium point→ structure of the coupling network:
  - Time-correlated noise: MT, T.Coletta, P.Jacquod Phys. rev. lett. **120**(8), 084101 (2018); MT, L.Pagnier, P.Jacquod Sci. adv. **5**(11), eaaw8359 (2019).
  - System-specific correlations: MT J. Phys: Complex. **3**(3), 03LT01 (2022); MT Chaos **32** (12) (2022).
  - Heterogeneous and correlated noise: J.Hindes, I.B.Schwartz arxiv:2023.13434 (2023).

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So far

- Linearization around an equilibrium point → structure of the coupling network:
- Transitions between basins of attraction:
  - Time-correlated noise: MT, R.Delabays, P.Jacquod Phys. Rev. E **99**(6), 062213.
  - System-specific correlations: MT Chaos **32** (12) (2022).
  - Heterogeneous and correlated noise: J.Hindes, I.B.Schwartz arxiv:2023.13434 (2023).

Conclusion and future work

So far

- Linearization around an equilibrium point→ structure of the coupling network:
- Transitions between basins of attraction:
  - Time-correlated noise: MT, R.Delabays, P.Jacquod Phys. Rev. E **99**(6), 062213.
  - System-specific correlations: MT Chaos **32** (12) (2022).
  - Heterogeneous and correlated noise: J.Hindes, I.B.Schwartz arxiv:2023.13434 (2023).

Future work

- Weak coupling and mobile oscillators.