

Tutorial on Synchronization I: Small and large fluctuations

Melvyn Tyloo

Director's Postdoc Fellow, T-4 and CNLS



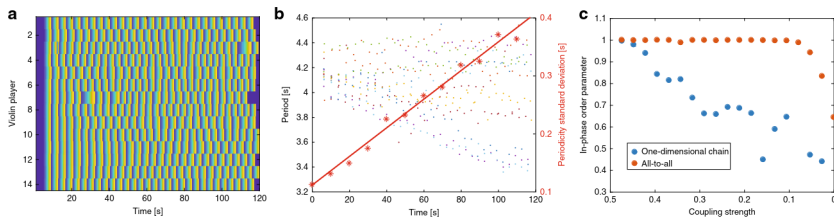
website: melvyntyloo.com

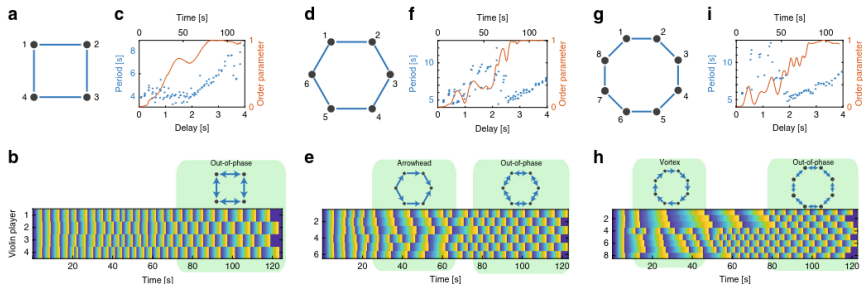
mtyloo@lanl.gov — 7/11/23

- Simple experiment: coupled metronomes

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- Other examples:
 - Fireflies flashing in unison
 - People clapping their hands or walking on a bridge
 - Synchronization of the quantum phase of Josephson junctions arrays
 - Synchronization of the phase of the voltages in electric power grids
 - Consensus formation on influence networks
 - Vehicular platoon formation
 - Orchestra



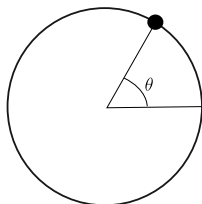




Shahal et al. Nature communications **11**, 3854 (2020).

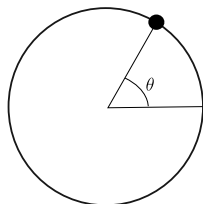
Modelling Synchronization

Single phase oscillator: $\dot{\theta} = \omega$

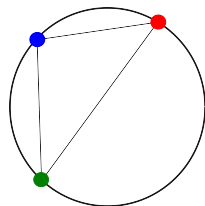


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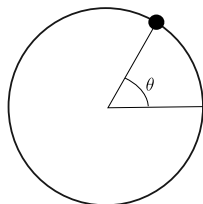


Coupled phase oscillators: $\dot{\theta}_i = \omega_i - \sum_j a_{ij} f(\theta_i - \theta_j)$

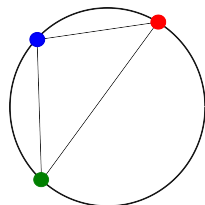


Modelling Synchronization

Single phase oscillator: $\dot{\theta} = \omega$



Coupled phase oscillators: $\dot{\theta}_i = \omega_i - \sum_j a_{ij} f(\theta_i - \theta_j)$



Synchronization: phase-locked $\dot{\theta}_i(t) = \dot{\theta}_j(t), \forall i, j$.

Modelling of Synchronization

Kuramoto model

$$\dot{\theta}_i = \omega_i - \frac{K}{N} \sum_{j=1}^N \sin(\theta_i - \theta_j), \text{ for } i = 1, \dots, N. \quad (1)$$

ω_i : natural frequencies.

Y. Kuramoto, Lecture Notes in Physics 39, International Symposium on Mathematical Problems in Theoretical Physics (1975).

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Order parameter

$$re^{i\psi} = N^{-1} \sum_{j=1}^N e^{i\theta_j} \quad (2)$$

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ψ : average phase.

Illustration

$$\dot{\theta}_i = \omega_i - Kr \sin(\psi - \theta_i), \text{ for } i = 1, \dots, N. \quad (3)$$

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Kuramoto model

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^N a_{ij} \sin(\theta_i - \theta_j), \text{ for } i = 1, \dots, N. \quad (4)$$

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Multistability

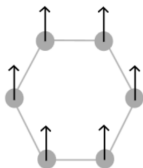
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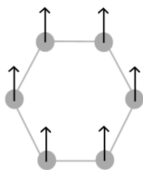
Synchronization on networks

Kuramoto model

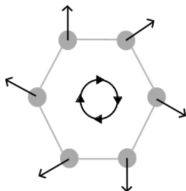
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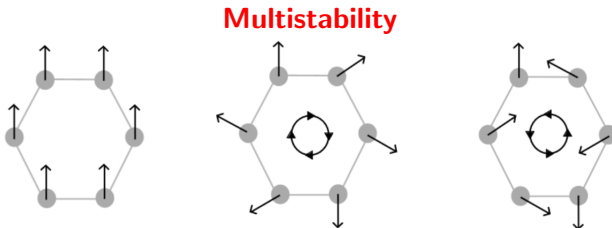


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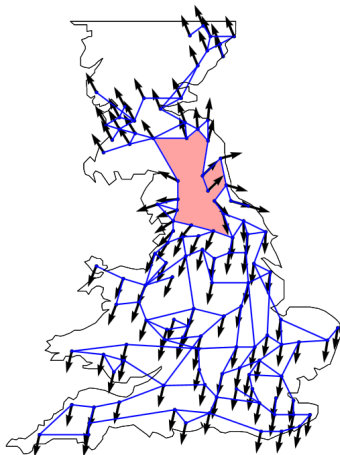
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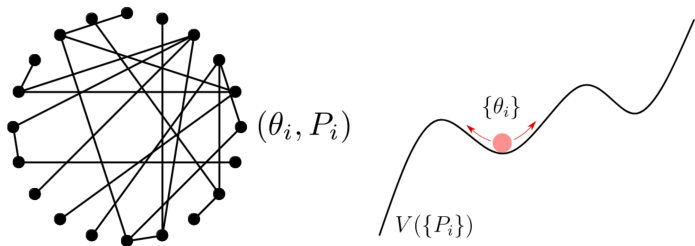


Synchronization on networks

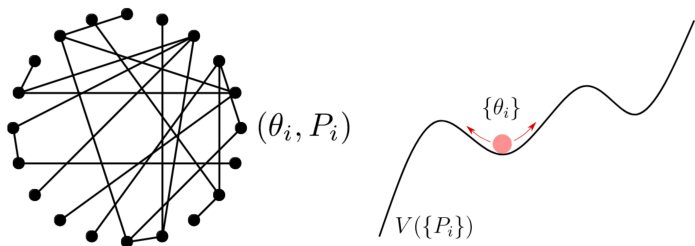


Delabays, MT, Jacquod, Chaos **27**(10), 103109 (2017)

Robustness of synchronous networks

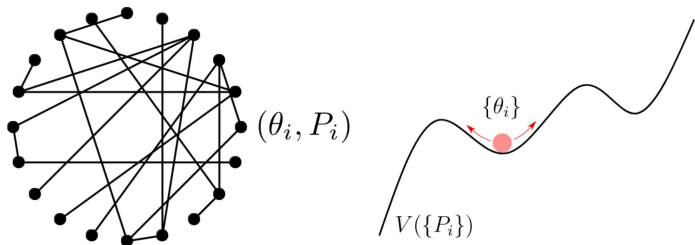


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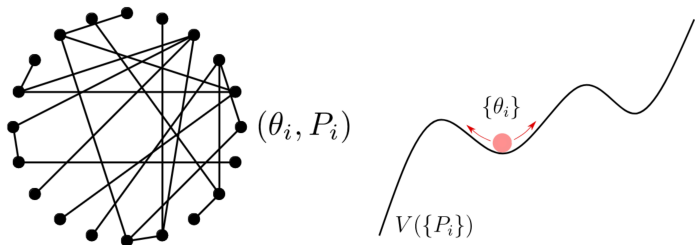
- Size of the basin of attraction

Robustness of synchronous networks



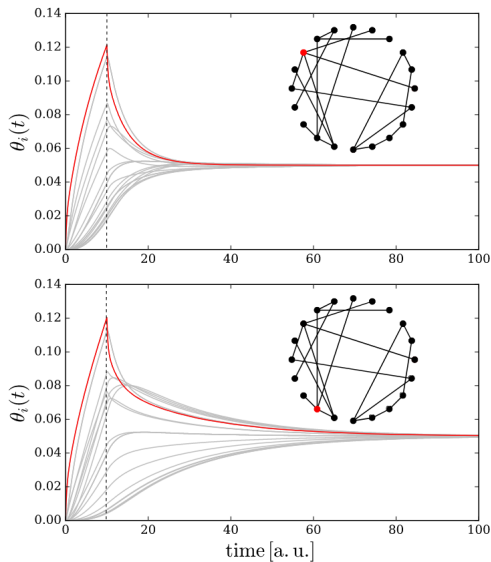
- Size of the basin of attraction
- Near equilibrium dynamics

Robustness of synchronous networks



- Size of the basin of attraction
- Near equilibrium dynamics
- Transitions between fixed points

Robustness of synchronous networks



Near equilibrium dynamics

$$0 = \omega_i - \sum_{j=1}^N a_{ij} \sin(\theta_i^{(0)} - \theta_j^{(0)}), \text{ for } i = 1, \dots, N. \quad (5)$$

Near equilibrium dynamics

$$0 = \omega_i - \sum_{j=1}^N a_{ij} \sin(\theta_i^{(0)} - \theta_j^{(0)}), \text{ for } i = 1, \dots, N. \quad (5)$$

$$\delta \dot{\theta}_i = - \sum_{j=1}^N a_{ij} \cos(\theta_i^{(0)} - \theta_j^{(0)}) (\delta \theta_i - \delta \theta_j) + \sum_m G_{im} \xi_m(t). \quad (6)$$

$\xi_i(t)$: Noise inputs.

Near equilibrium dynamics

$$\delta\dot{\theta} = -\mathbb{L}(\{\theta_k^{(0)}\})\delta\theta + \mathbf{G}\xi(t), \text{ for } i = 1, \dots, N. \quad (7)$$

$\mathbb{L}(\{\theta_k^{(0)}\})$: Jacobian, with eigenvalues $\lambda_1 = 0 < \lambda_2 \leq \dots \leq \lambda_N$.

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Solution

$$\delta\theta_i(t) = \sum_{\alpha} e^{-\lambda_{\alpha} t} \int_0^t e^{\lambda_{\alpha} t'} \sum_{j,m} G_{mj} \xi_j(t') u_{\alpha,m} dt' u_{\alpha,i}. \quad (8)$$

Robustness of synchronous networks to noise inputs

Uncorrelated white noise

$$\langle \xi_i(t) \xi_j(t') \rangle = \xi_0^2 \tau_0 \delta_{ij} \delta(t - t') \quad (9)$$

$$G_{ij} = \delta_{ij} \quad (10)$$

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Variance at node i

$$\langle \delta\theta_i^2 \rangle = \frac{\xi_0^2 \tau_0}{2} \sum_{\alpha} \frac{u_{\alpha,i}^2}{\lambda_{\alpha}} \quad (11)$$

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Average variance in the network

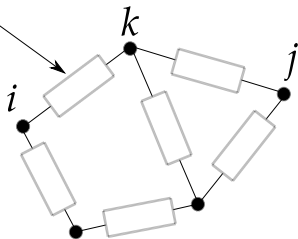
$$N^{-1} \sum_i \langle \delta\theta_i^2 \rangle = \frac{\xi_0^2 \tau_0}{2N} \sum_{\alpha} \frac{1}{\lambda_{\alpha}} \quad (12)$$

Resistance Distance

$$\Omega_{ij}^{(1)} = \mathbb{L}_{ii}^\dagger + \mathbb{L}_{jj}^\dagger - \mathbb{L}_{ij}^\dagger - \mathbb{L}_{ji}^\dagger = \sum_{\alpha \geq 2} \frac{(u_{\alpha,i} - u_{\alpha,j})^2}{\lambda_\alpha}.$$

\mathbb{L}^\dagger : pseudo inverse of \mathbb{L} (because of $\lambda_1 = 0$).

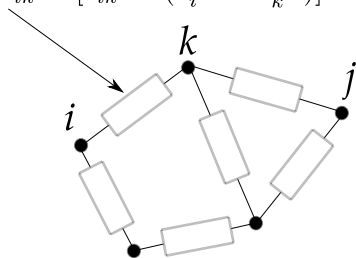
$$R_{ik} = [a_{ik} \cos(\theta_i^{(0)} - \theta_k^{(0)})]^{-1}$$



Kirchhoff Index

$$Kf_1 = \sum_{i < j} \Omega_{ij}^{(1)} = n \sum_{\alpha \geq 2} \lambda_{\alpha}^{-1} .$$

$$R_{ik} = [a_{ik} \cos(\theta_i^{(0)} - \theta_k^{(0)})]^{-1}$$



Generalized Resistance Distances

$$\begin{aligned}\Omega_{ij}^{(p)} &= \mathbb{L}'_{ii} + \mathbb{L}'_{jj} - \mathbb{L}'_{ij} - \mathbb{L}'_{ji} \\ &= \sum_{\alpha \geq 2} \frac{(u_{\alpha,i} - u_{\alpha,j})^2}{\lambda_{\alpha}^p}, \\ \mathbb{L}' &= \mathbb{L}^p.\end{aligned}$$

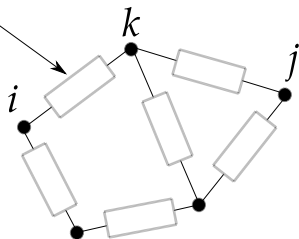
Generalized Kirchhoff Indices

$$Kf_p = \sum_{i < j} \Omega_{ij}^{(p)} = n \sum_{\alpha \geq 2} \lambda_{\alpha}^{-p}.$$

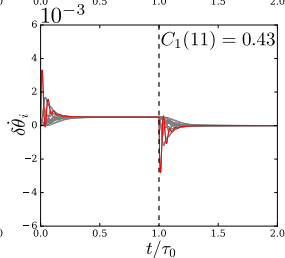
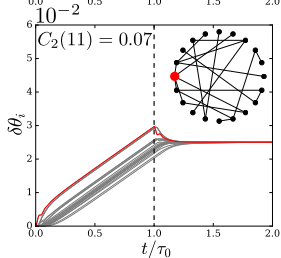
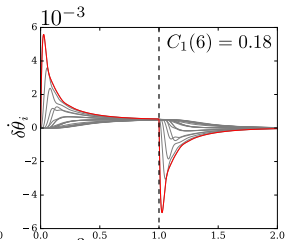
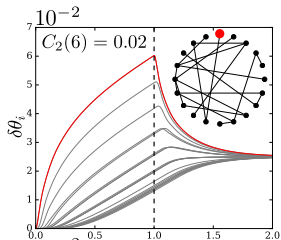
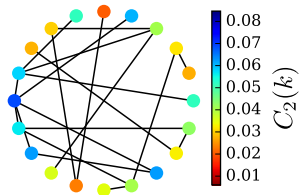
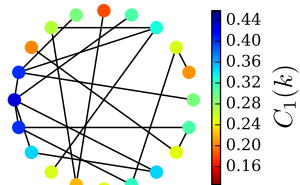
Centralities

$$C_p(k) = \left[n^{-1} \sum_j \Omega_{kj}^{(p)} \right]^{-1} = \left[\sum_{\alpha \geq 2} \frac{u_{\alpha,k}^2}{\lambda_\alpha^p} + n^{-2} Kf_p \right]^{-1}.$$

$$R_{ik} = [a_{ik} \cos(\theta_i^{(0)} - \theta_k^{(0)})]^{-1}$$



Local Vulnerabilities and C_p 's: Numerics



Indicator Mode Approximation

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^N A_{ij} \sin(\theta_i - \theta_j) + \sum_m G_{im} \xi_m(t), \text{ for } i = 1, \dots, N. \quad (13)$$

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$$\theta_i(t) = \theta_i^* + a(t) \Delta \theta_i \quad \forall i, \text{ where } \Delta \theta_i \equiv \theta_i^s - \theta_i^*$$

$$\dot{a} = F(a) + \frac{\sum_i \Delta \theta_i \sum_m G_{im} \xi_m(t)}{\sum_l \Delta \theta_l^2}, \quad \text{where}$$
$$F(a) = \frac{\sum_i \Delta \theta_i \left[\omega_i + K \sum_j A_{ij} \sin(\theta_j^* - \theta_i^* + a(\Delta \theta_j - \Delta \theta_i)) \right]}{\sum_l \Delta \theta_l^2}. \quad (14)$$

Indicator Mode Approximation

$$\langle \eta(t)\eta(t') \rangle = \sigma^2 \sum_m Q_m^2 \delta(t - t'), \quad \text{where}$$

$$Q_m = \frac{\sum_i G_{im} \Delta\theta_i}{\sum_l \Delta\theta_l^2}. \quad (15)$$

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Fokker-Planck equation for the probability flux of a ,

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial a} [F(a)P] + \frac{1}{2} \frac{\partial^2}{\partial a^2} [\sigma^2 \sum_m Q_m^2 P]. \quad (16)$$

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WKB form: $P \sim \exp[-2S(a, t)/\sigma^2]$.

Indicator Mode Approximation

$\partial S / \partial t = 0$. In this case $H=0$, and therefore

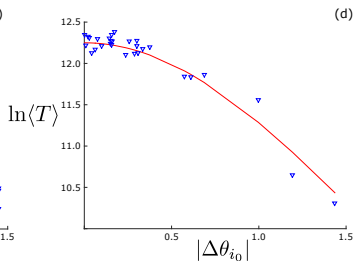
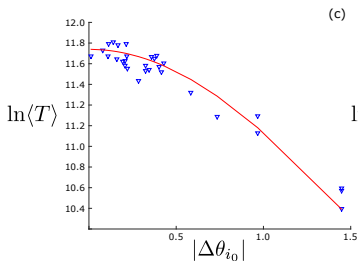
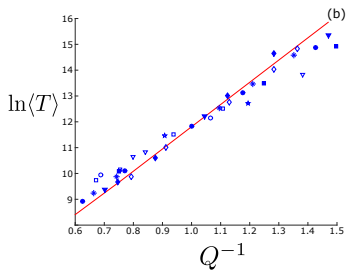
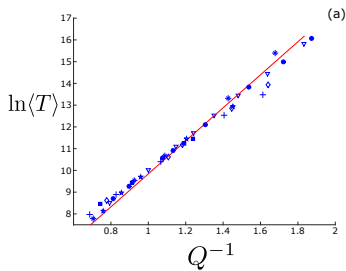
$$\frac{\partial S}{\partial a} = \frac{dS}{da} = -\frac{F(a)}{\sum_m Q_m^2}. \quad (17)$$

$$S = \frac{\sum_l \Delta\theta_l^2}{\sum_{m,i} (G_{mi} \Delta\theta_i)^2} S_{DB}, \quad \text{where}$$

$$S_{DB} = \sum_i \Delta\theta_i \left[-\omega_i + K \sum_{\substack{j, \\ \Delta\theta_i \neq \Delta\theta_j}} \frac{A_{ij}}{\Delta\theta_j - \Delta\theta_i} \cdot (\cos(\theta_j^s - \theta_i^s) - \cos(\theta_j^* - \theta_i^*)) \right]. \quad (18)$$

$$Q = \frac{\Delta\theta^T G G^T \Delta\theta}{\Delta\theta^T \Delta\theta}. \quad (19)$$

Indicator Mode Approximation



So far

- Linearization around an equilibrium point → structure of the coupling network:
 - Time-correlated noise: MT, T.Coletta, P.Jacquod Phys. rev. lett. **120**(8), 084101 (2018); MT, L.Pagnier, P.Jacquod Sci. adv. **5**(11), eaaw8359 (2019).
 - System-specific correlations: MT J. Phys: Complex. **3**(3), 03LT01 (2022); MT Chaos **32** (12) (2022).
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Future work

- Weak coupling and mobile oscillators.