Network response and reconstruction

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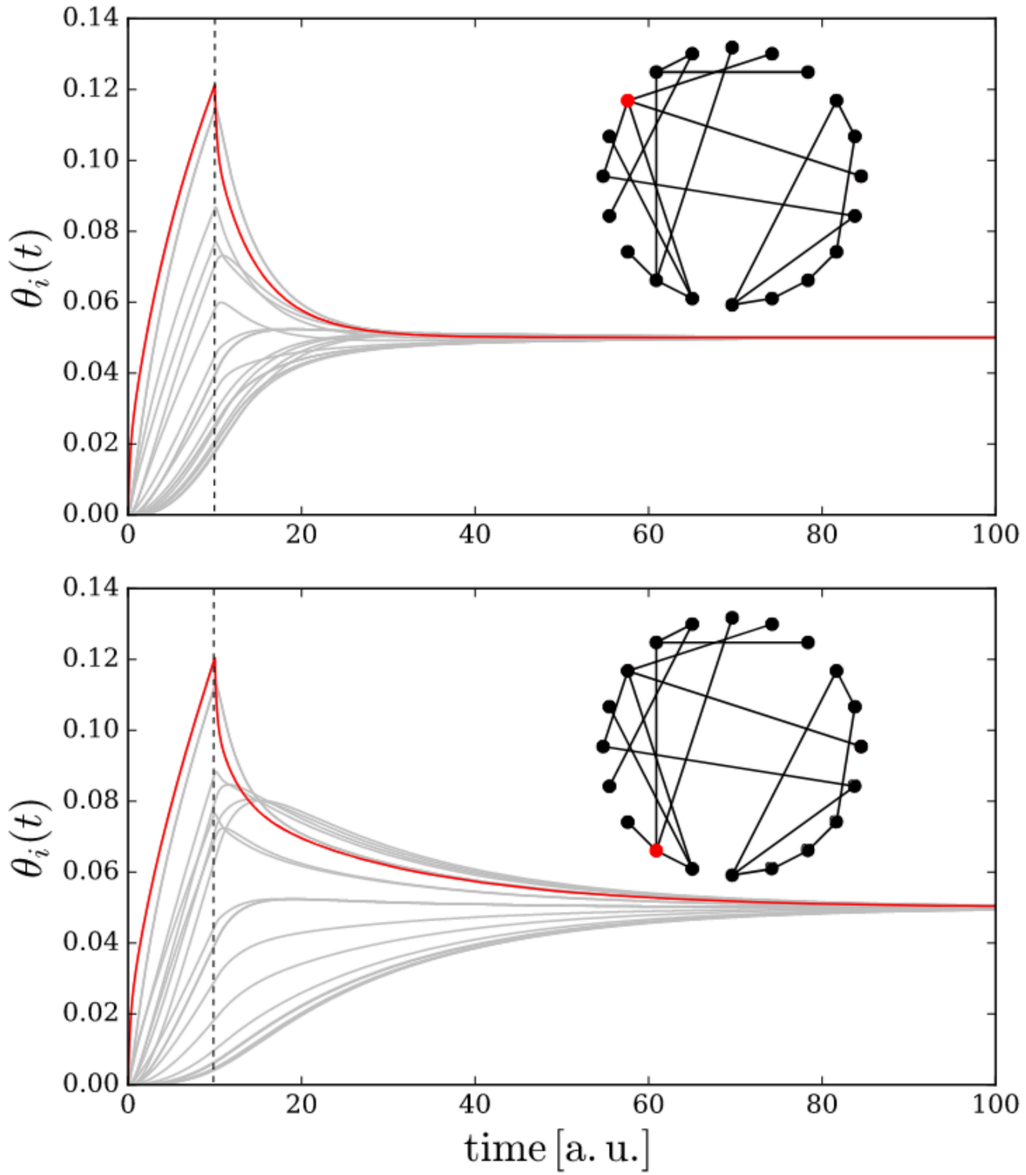
*MT, Phys. Rev. E **112**, L042301 (2025)*
*MT, EPL **156**, 51001 (2026)*

Center for Complex Systems, Queen Mary University — 26/11/2026

Resilience of networked systems

- Dynamics and structure:



Resilience of networked systems

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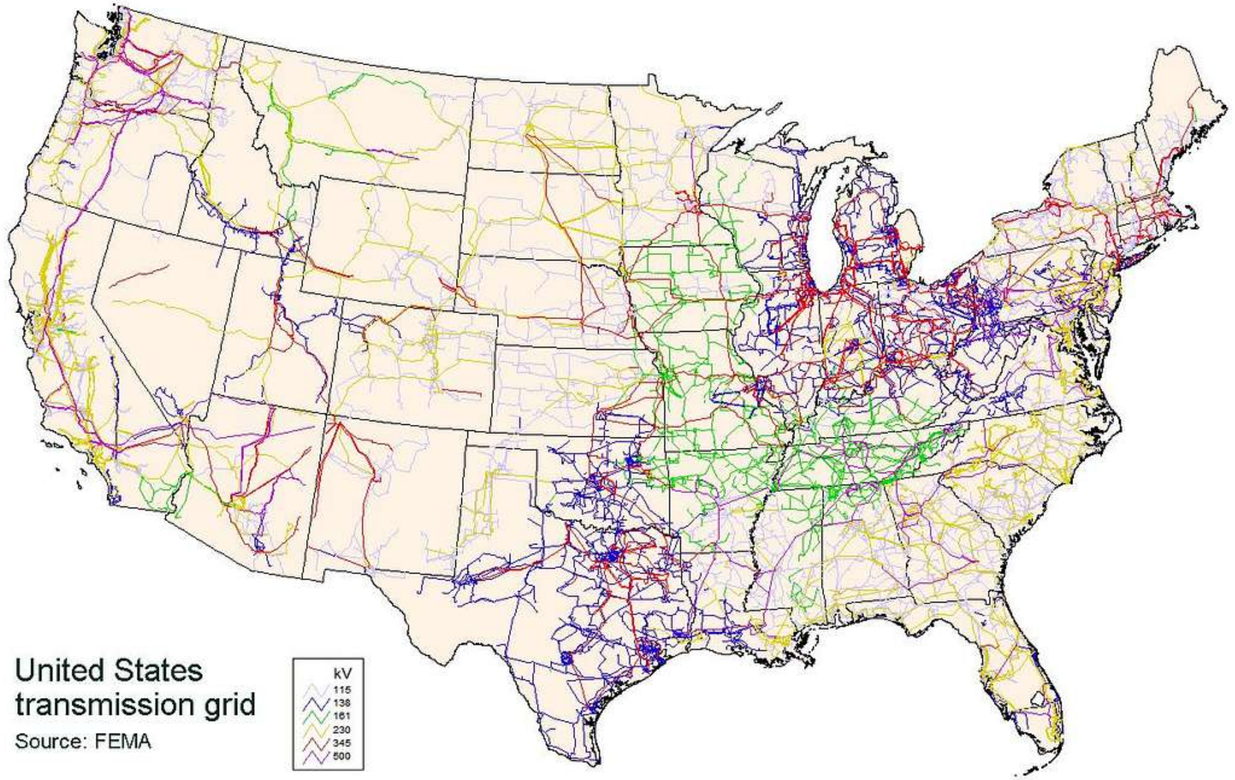

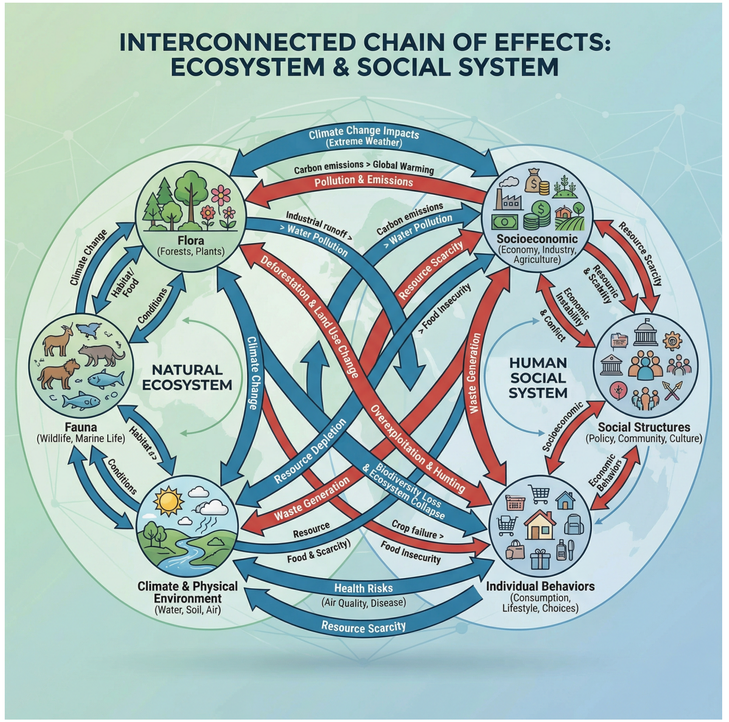

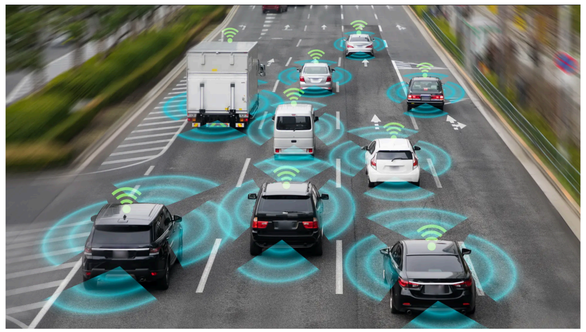

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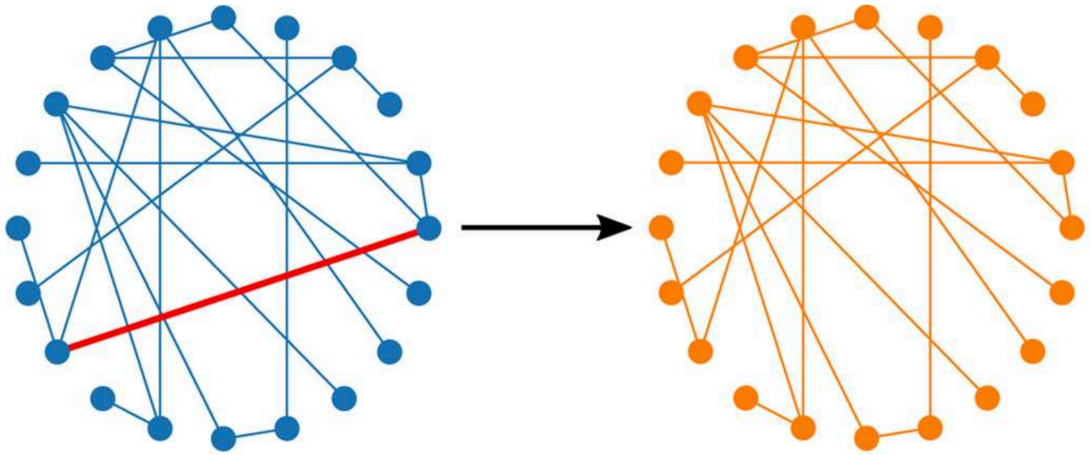
Resilience of networked systems

- Dynamics and structure:
  - How does the response change when the structure changes?



Resilience of networked systems

- Dynamics and structure:
  - How does the response change when the structure changes?
    - ★ Symmetry of the interaction/perturbation.
    - ★ Normality of the interaction/perturbation.
    - ★ Set of matrices that identify the most impacted nodes.

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Networked system

Simple linear system:

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \eta$$

$\mathbf{x} \in \mathbb{R}^N$	State of the system.
$\mathbf{A} \in \mathbb{R}^{N \times N}$	Coupling between the degrees of freedom, assumed symmetric.
$\eta \in \mathbb{R}^N$	External input.

Networked system

Linear approximation of the dynamics around a fixed point:

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \eta \qquad \dot{\mathbf{y}} = \mathbf{F}(\mathbf{y}) + \eta\,, \quad A_{ij} = \frac{\partial F_i}{\partial y_j}\Big|_{\mathbf{y}=\mathbf{y}^*}$$

$\mathbf{x} \in \mathbb{R}^N$	Deviation from the fixed point, i.e. $\mathbf{x} = \mathbf{y} - \mathbf{y}^*$
$\mathbf{A} \in \mathbb{R}^{N \times N}$	Jacobian matrix.
$\eta \in \mathbb{R}^N$	External input.

Networked system

Linear approximation of the dynamics around a fixed point:

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \eta$$

Solution:

$$\mathbf{x}(t) = \int_0^t e^{\mathbf{A}\,(t-t')}\eta(t')\,\mathrm{d}t' \,.$$

Averaged response

Linear approximation of the dynamics around a fixed point:

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \eta$$

Solution:

$$\mathbf{x}(t) = \int_0^t e^{\mathbf{A}\,(t-t')}\eta(t')\,\mathrm{d}t'$$

$$\mathbf{x}(i,t) = [e^{\mathbf{A}t}]_{.,i}$$

$$\eta(i,t) = \hat{\mathbf{e}}_i\,\delta(t)\,,\, i = 1,\ldots,N$$

Averaged response

Linear approximation of the dynamics around a fixed point:

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \eta$$

$$\overline{x_j^2}(t) = \sum_{i=1}^{N} [\mathbf{x}(i,t)\mathbf{x}(i,t)^\top]_{jj} = [e^{\mathbf{A}t} e^{\mathbf{A}^\top t}]_{jj} = [e^{2\mathbf{A}t}]_{jj}$$

Structural perturbation

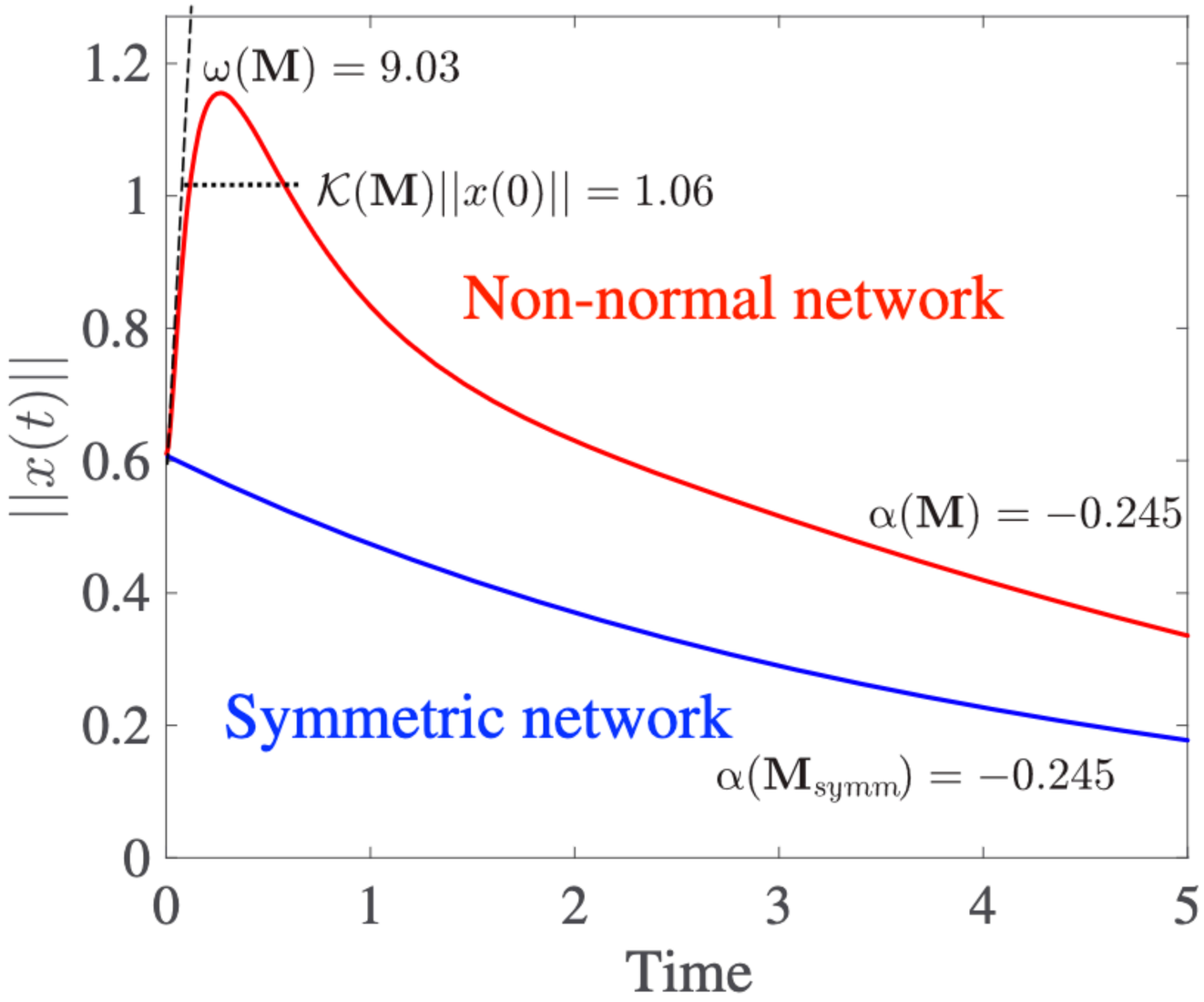
Perturbation of the dynamics:

$$\dot{\mathbf{x}} = (\mathbf{A} + \Delta)\,\mathbf{x} + \eta \qquad \Delta \text{ Structural alteration}$$

$$\overline{x_j^2}(t) = \sum_{i=1}^{N} [\mathbf{x}(i,t)\mathbf{x}(i,t)^\top]_{jj} = [e^{(\mathbf{A}+\boldsymbol{\Delta})t} e^{(\mathbf{A}^\top+\boldsymbol{\Delta}^\top)t}]_{jj}$$

Non-normality $[\mathbf{A} + \boldsymbol{\Delta}, \mathbf{A}^\top + \boldsymbol{\Delta}^\top] \neq 0$

(Non-normality)



M. Asllani et al. *Science advances* 4.12 (2018): eaau9403.

Structural perturbation

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Non-normality $[\mathbf{A}+\mathbf{\Delta}, \mathbf{A}^\top+\mathbf{\Delta}^\top] \neq 0$

$$\overline{x_j^2}(t) = [e^{\mathbf{B}\,t}]_{jj}$$

$$\mathbf{B} = (\mathbf{A}+\Delta)t + (\mathbf{A}+\Delta)^\top t + \frac{1}{2}[\mathbf{A}+\Delta, (\mathbf{A}+\Delta)^\top]t^2 + \frac{1}{12}[\mathbf{A}+\Delta, [\mathbf{A}+\Delta, (\mathbf{A}+\Delta)^\top]]t^3 + \dots$$

Baker–Campbell–Hausdorff formula

Structural perturbation

Perturbation of the dynamics:

$$\overline{x_j^2}(t) = \sum_{i=1}^{N} [\mathbf{x}(i,t)\mathbf{x}(i,t)^\top]_{jj} = [e^{(\mathbf{A}+\mathbf{\Delta})t} e^{(\mathbf{A}^\top+\mathbf{\Delta}^\top)t}]_{jj}$$

Non-normality $[\mathbf{A}+\mathbf{\Delta}, \mathbf{A}^\top+\mathbf{\Delta}^\top] \neq 0$

$$\overline{x_j^2}(t) = [e^{\mathbf{M}_1 t + \mathbf{M_2}\frac{t^2}{2} + \mathbf{M_3}\frac{t^3}{12} + h.o.t.}]_{jj}$$

$$\mathbf{M}_1 = 2\mathbf{A} + \mathbf{\Delta} + \mathbf{\Delta}^\top,$$

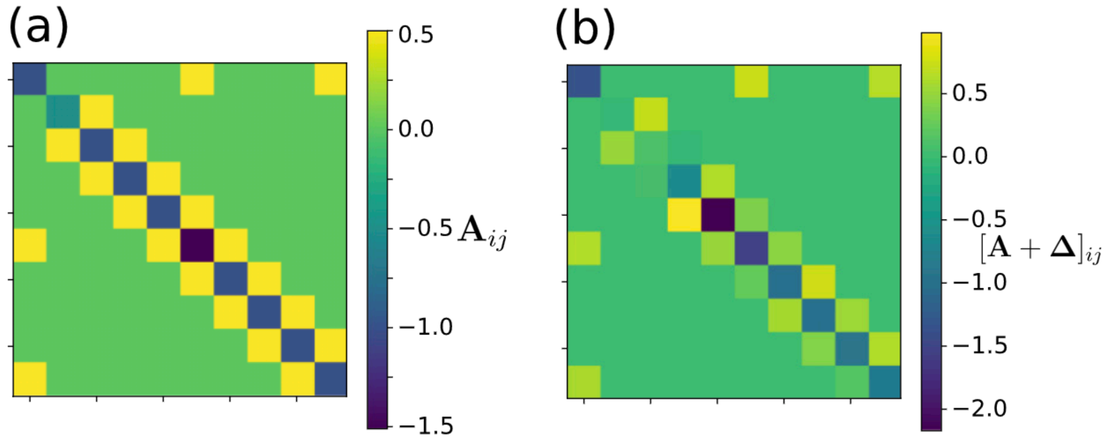
$$\mathbf{M}_2 = [\mathbf{A}, \mathbf{\Delta}^\top - \mathbf{\Delta}] + [\mathbf{\Delta}, \mathbf{\Delta}^\top],$$

$$\mathbf{M}_3 = [\mathbf{\Delta} - \mathbf{\Delta}^\top, [\mathbf{A}, \mathbf{\Delta}^\top - \mathbf{\Delta}] + [\mathbf{\Delta}, \mathbf{\Delta}^\top]\,].$$

Global response

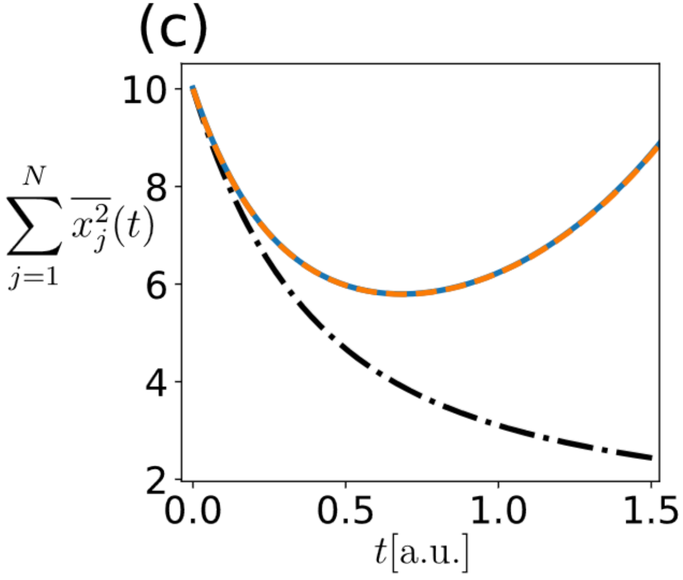
Approximation:

$$\overline{x_j^2}(t) \cong [e^{\mathbf{M}_1 t + \mathbf{M}_2 \frac{t^2}{2} + \mathbf{M}_3 \frac{t^3}{12}}]_{jj}$$



Network level response:

$$\sum_{j=1}^{N} \overline{x_j^2}(t) \cong \mathrm{Tr}\left[e^{\mathbf{M}_1 t + \mathbf{M}_2 \frac{t^2}{2} + \mathbf{M}_3 \frac{t^3}{12}}\right] .$$



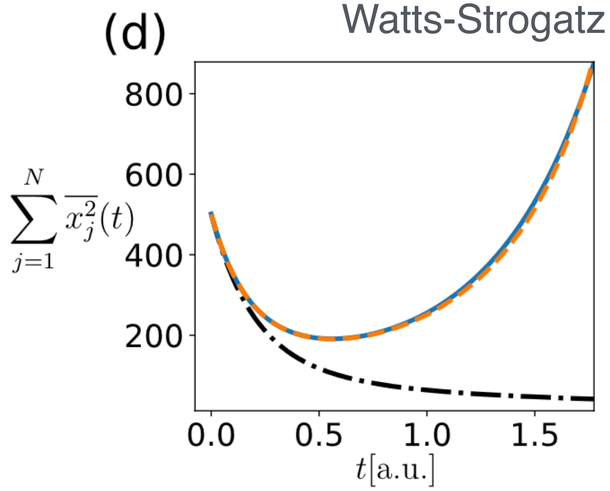
Global response

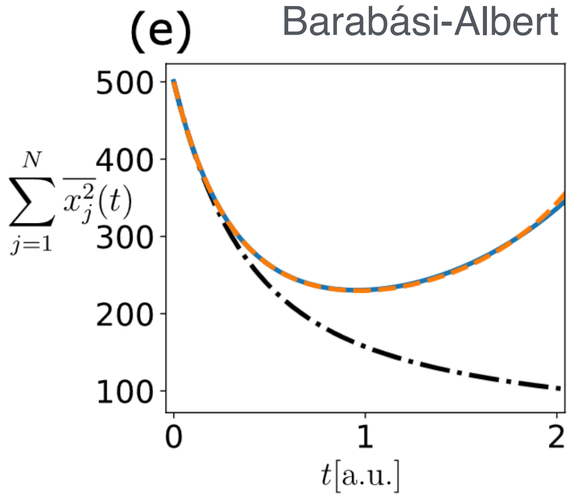
Approximation:

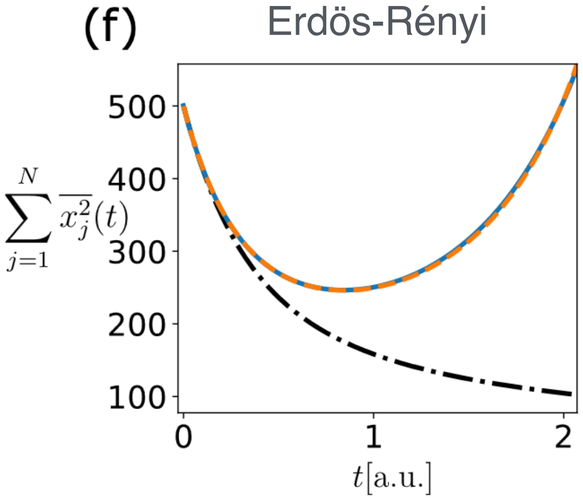
$$\overline{x_j^2}(t) \cong \left[e^{\mathbf{M}_1 t + \mathbf{M_2}\frac{t^2}{2} + \mathbf{M_3}\frac{t^3}{12}}\right]_{jj}$$

Network level response:

$$\sum_{j=1}^{N} \overline{x_j^2}(t) \cong \mathrm{Tr}\left[e^{\mathbf{M}_1 t + \mathbf{M_2}\frac{t^2}{2} + \mathbf{M_3}\frac{t^3}{12}}\right]$$



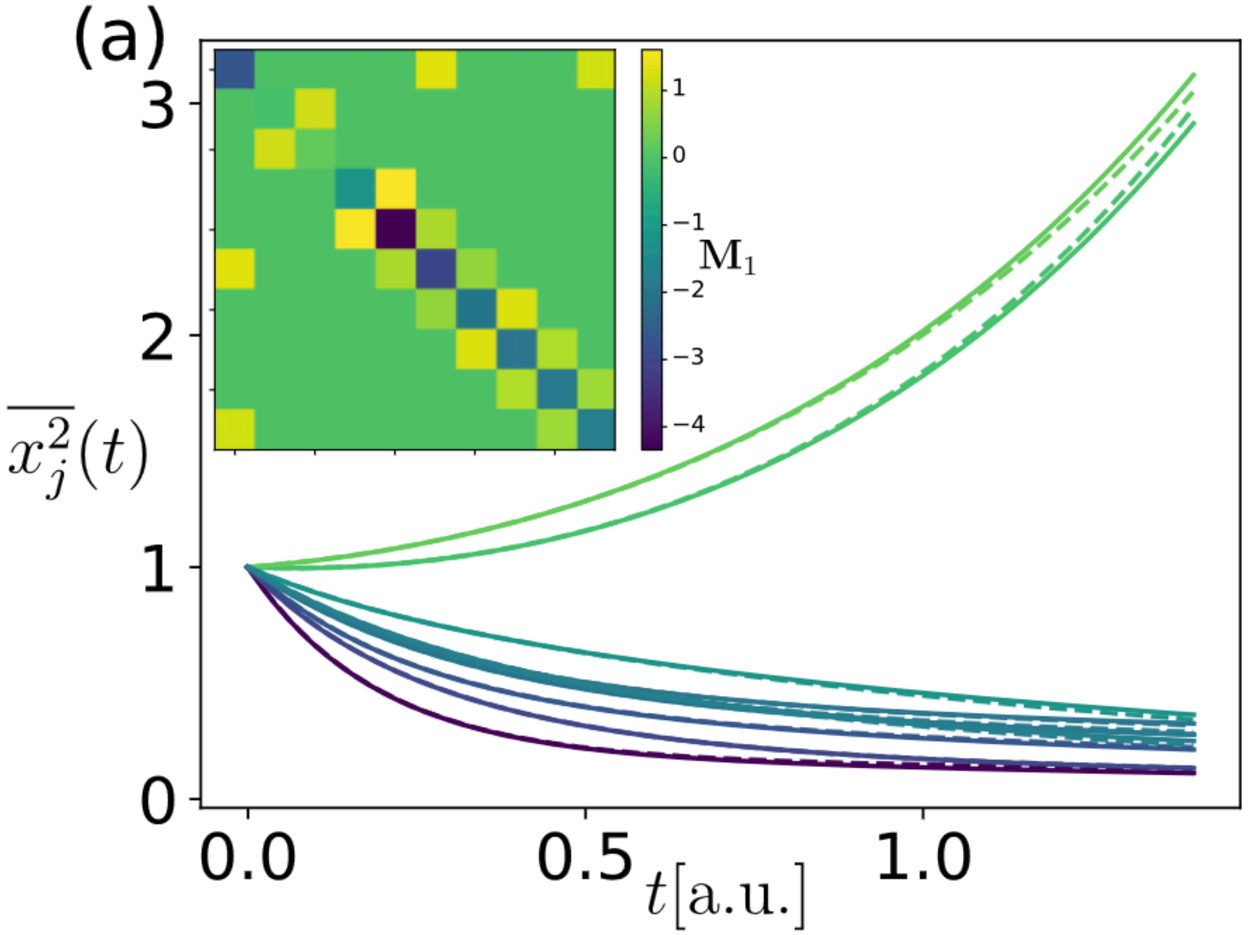




Local response

Node level response:

$$\overline{x_j^2}(t) \cong [e^{\mathbf{M}_1 t + \mathbf{M_2} \frac{t^2}{2} + \mathbf{M_3} \frac{t^3}{12}}]_{jj}$$

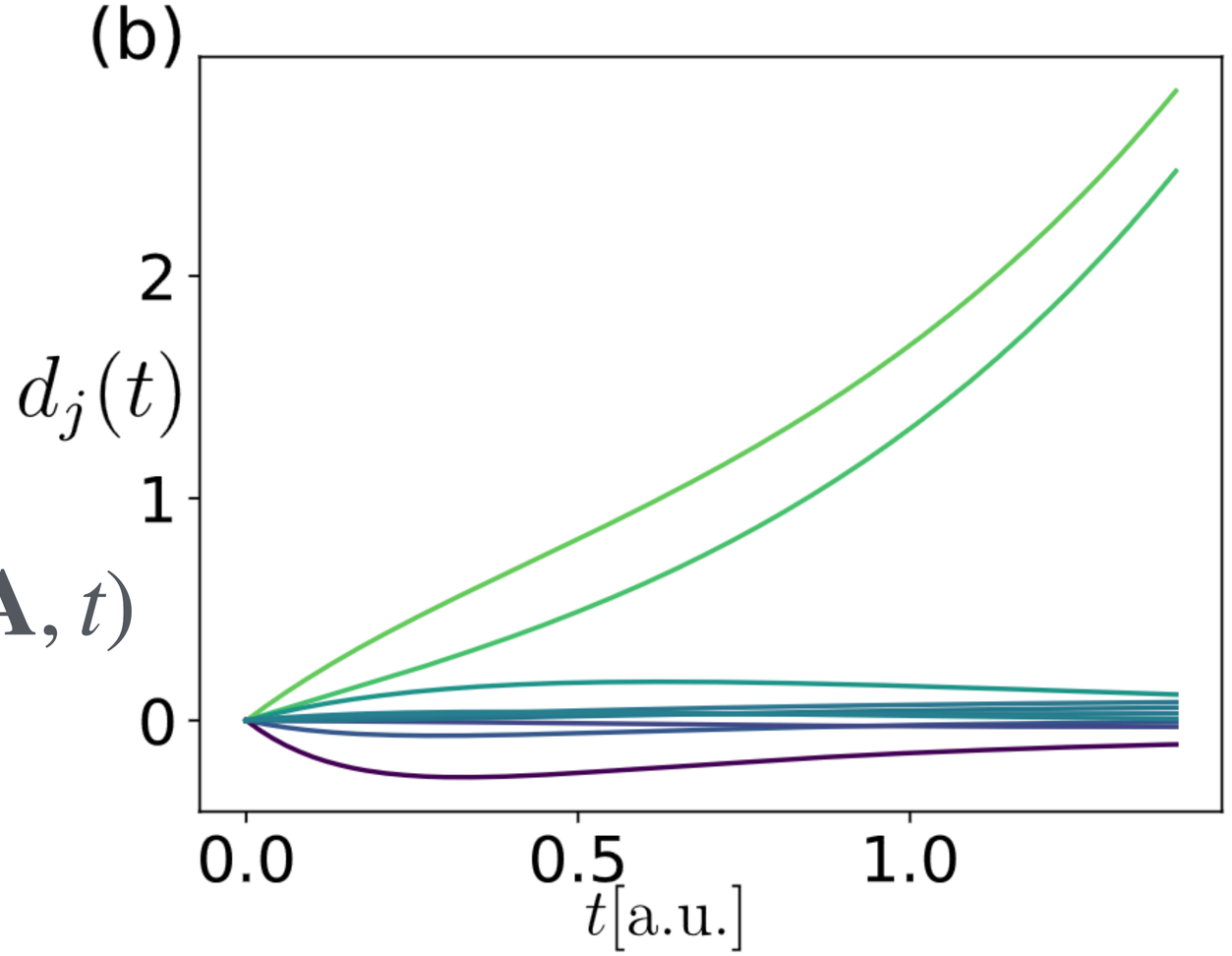


Local response

Node level response:

$$\overline{x_j^2}(t) \cong [e^{\mathbf{M}_1 t + \mathbf{M}_2 \frac{t^2}{2} + \mathbf{M}_3 \frac{t^3}{12}}]_{jj}$$

$$d_j(t) = \overline{x_j^2}(\mathbf{A} + \mathbf{\Delta}, t) - \overline{x_j^2}(\mathbf{A}, t)$$



Global vs. Local response

Network level response:

$$\sum_{j=1}^{N} \overline{x_j^2}(t) \cong N + \mathrm{Tr}[\mathbf{M}_1]\, t + \mathrm{Tr}[\mathbf{M}_1^2]\, \frac{t^2}{2} + \mathrm{Tr}[\mathbf{M}_1^3]\, \frac{t^3}{6}$$

$$+ \left(\mathrm{Tr}[\mathbf{M}_1^4] + 2\,\mathrm{Tr}[\mathbf{M}_1\mathbf{M}_3] + 3\,\mathrm{Tr}[\mathbf{M}_2^2]\right)\, \frac{t^4}{24} + \mathcal{O}(t^5)$$

Node level response:

$$\overline{x_j^2}(t) \cong 1 + [\mathbf{M}_1]_{jj}\, t + \left([\mathbf{M}_1^2]_{jj} + [\mathbf{M}_2]_{jj}\right)\, \frac{t^2}{2}$$

$$+ \left([\mathbf{M}_1^3]_{jj} + 3[\mathbf{M}_1\mathbf{M}_2]_{jj} + \frac{1}{2}[\mathbf{M}_3]_{jj}\right)\, \frac{t^3}{6} + \mathcal{O}(t^4)$$

Global vs. Local response

Network level

$$\sum_{j=1}^{N} \overline{x_j^2}(t) \cong N + \mathrm{Tr}[\mathbf{M}_1]\, t + \mathrm{Tr}[\mathbf{M}_1^2]\, \frac{t^2}{2} + \mathrm{Tr}[\mathbf{M}_1^3]\, \frac{t^3}{6}$$

$$+ \left(\mathrm{Tr}[\mathbf{M}_1^4] + 2\,\mathrm{Tr}[\mathbf{M}_1\mathbf{M}_3] + 3\,\mathrm{Tr}[\mathbf{M}_2^2]\right)\, \frac{t^4}{24} + \mathcal{O}(t^5)$$

Node level

$$\overline{x_j^2}(t) \cong 1 + [\mathbf{M}_1]_{jj}\, t + \left([\mathbf{M}_1^2]_{jj} + [\mathbf{M}_2]_{jj}\right)\, \frac{t^2}{2}$$

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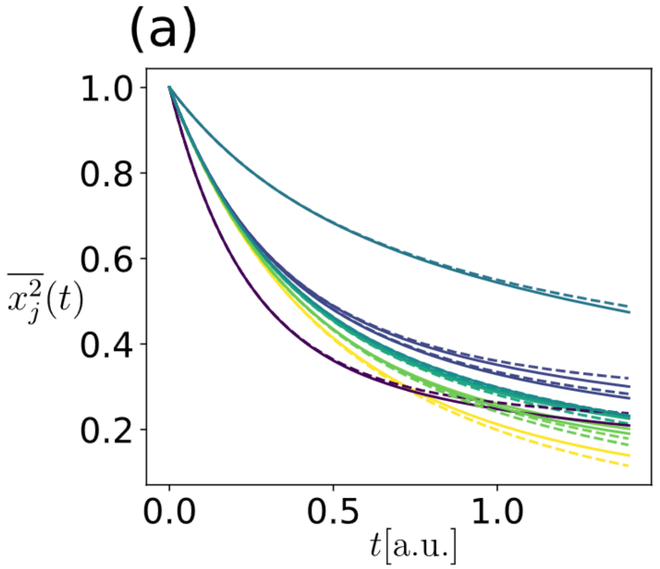
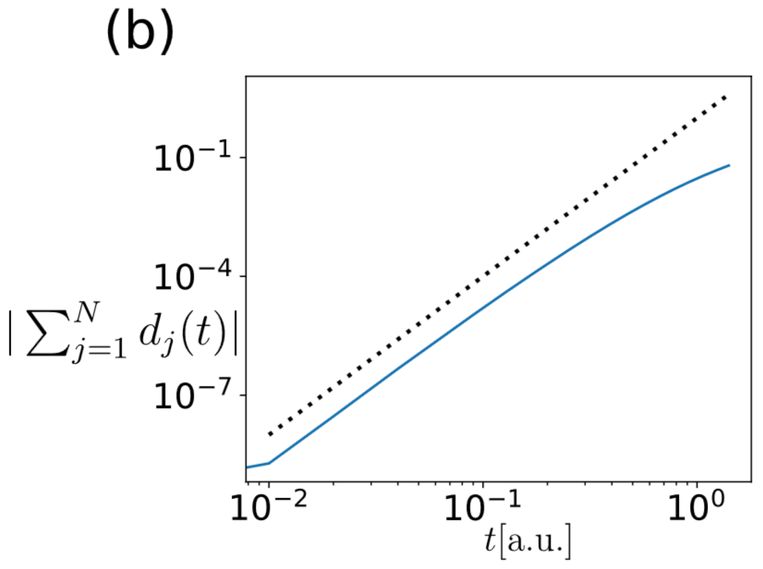
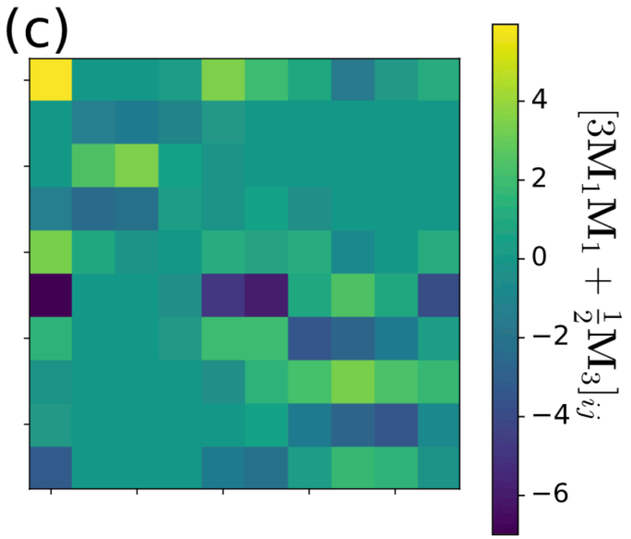
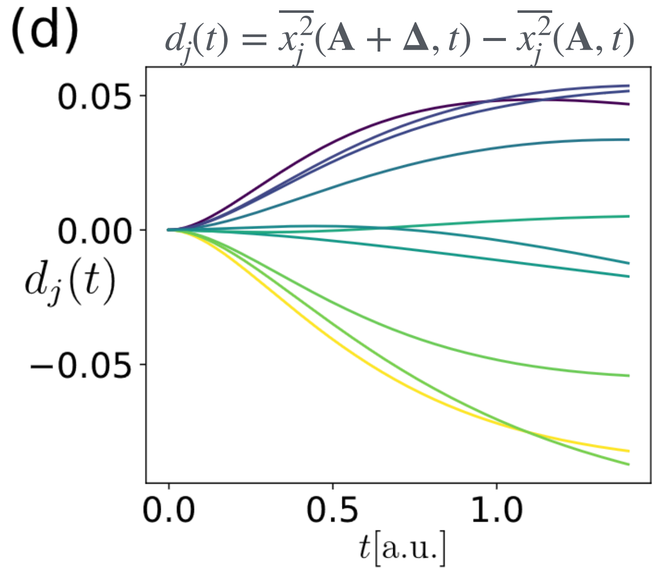
Skew-symmetric perturbation









Conclusion: Part 1

University of Exeter

- Set of matrices to identify impacted nodes.
- Perturbations not visible in global response but visible locally.

$$\overline{x_j^2}(t) = \left[e^{\mathbf{M}_1 t + \mathbf{M}_2 \frac{t^2}{2} + \mathbf{M}_3 \frac{t^3}{12} + h.o.t.}\right]_{jj}$$

$$\mathbf{M}_1 = 2\mathbf{A} + \boldsymbol{\Delta} + \boldsymbol{\Delta}^\top,$$

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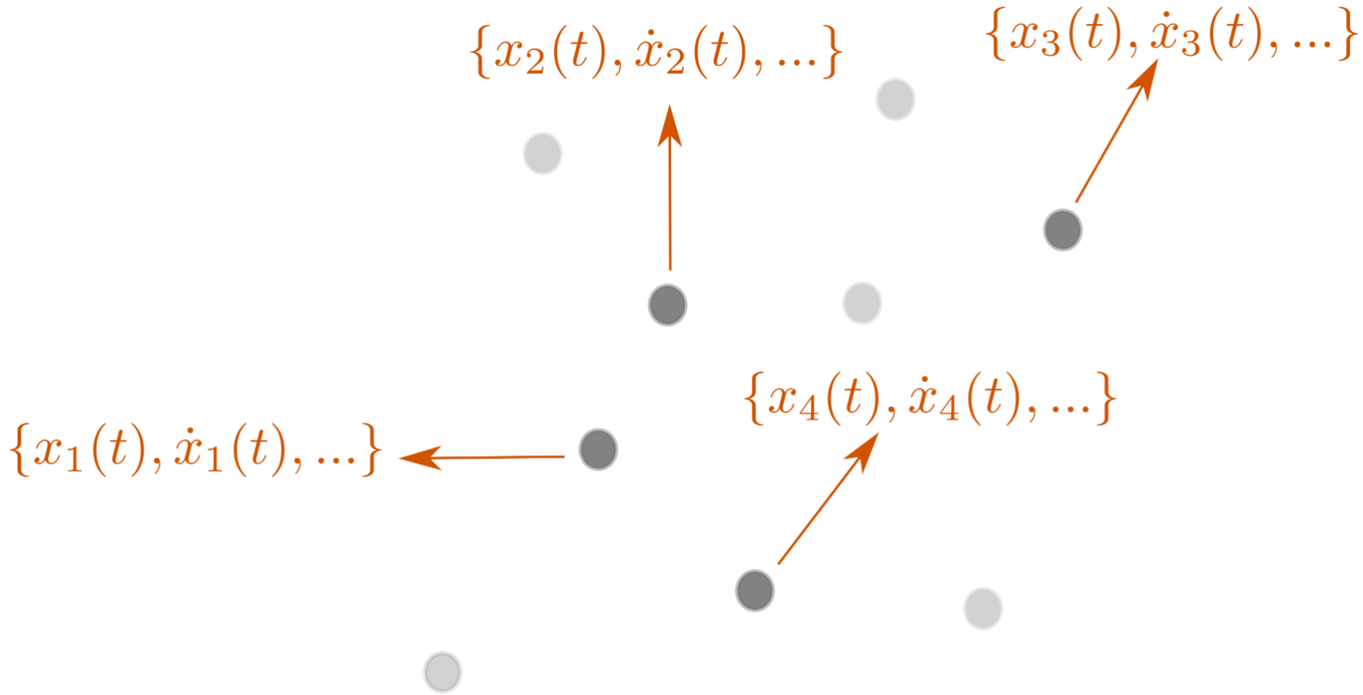
Network reconstruction

- Structure from observed dynamics:
  - ★ Model-free inference.
    - ✳ Functional network
  - ★ Model-based inference.
    - ✳ Structural network



*MT, EPL **156**, 51001 (2026)*

Network reconstruction

Leader-follower consensus dynamics

$$x_i(t+\Delta t) = x_i(t) - \sum_{j=1}^{N} k_{ij}\,[x_i(t) - x_j(t)] + \xi_i(t)\,, \quad \text{for } i = 1,...N_f\,, \text{ and,}$$

$$x_i(t+\Delta t) = \alpha_i x_i(t) - \sum_{j=1}^{N} k_{ij}\,[x_i(t) - x_j(t)]\,, \quad \text{for } i = N_f+1,...N\,.$$

Noise: $\langle \xi_i(t)\xi_j(t')\rangle = \xi_{i,0}^2 \delta_{ij}\,\delta_{tt'}\,.$

Network reconstruction

Leader-follower consensus dynamics

$$x_i(t+\Delta t) = x_i(t) - \sum_{j=1}^{N} k_{ij}\left[x_i(t) - x_j(t)\right] + \xi_i(t)\,, \quad \text{for } i = 1,...N_f\,, \text{ and,}$$

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$$\mathbf{x}(t+\Delta t) = \begin{bmatrix} \mathbf{x}_o(t+\Delta t) \\ \mathbf{x}_h(t+\Delta t) \end{bmatrix} = \underbrace{\begin{bmatrix} BC \\ DE \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} \mathbf{x}_o(t) \\ \mathbf{x}_h(t) \end{bmatrix} + \begin{bmatrix} \xi_o(t) \\ \mathbf{0} \end{bmatrix}.$$

Network reconstruction

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Some constraints on A

$$B_{ij} = \delta_{ij}(1 - \sum_{l=1}^{N} k_{il}) + k_{ij}\,, i,j = 1,...,N_f$$

$$\sum_{j=1;j\neq i}^{N_f} B_{ij} + \sum_{k=1}^{N_l} C_{ik} = -B_{ii}$$

$$E_{ij} = \delta_{ij}\alpha_i - \sum_{l=1}^{N_l} k_{i+N_f l+N_f} + k_{i+N_f j+N_f}\,, i,j = 1,...,N_l$$

$$\sum_{j=1}^{N_f} D_{ij} + \sum_{k=1}^{N_l} E_{ik} = \Lambda_{ii}$$

$$C_{ij} = k_{ij+N_f}\,, i = 1,...,N_f\,, j = 1,...,N_l$$

$$D_{ij} = k_{i+N_f j}\,, i = 1,...,N_l\,, j = 1,...,N_f$$

Network reconstruction

Leader-follower consensus dynamics

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Autoregressive expansion

$$\begin{aligned} \mathbf{x}_o(t+\Delta t) &= B\,\mathbf{x}_o(t) + \xi_o(t) + CD\,\mathbf{x}_o(t-\Delta t) + CED\,\mathbf{x}_o(t-2\Delta t) + CE^2D\,\mathbf{x}_o(t-3\Delta t) + \dots \\ &= B\,\mathbf{x}_o(t) + \xi_o(t) + \sum_{k=0}^{M} CE^kD\,\mathbf{x}_o(t-(k+1)\Delta t)\,, \end{aligned}$$

Network reconstruction

Autoregressive expansion

$$\mathbf{x}_o(t+\Delta t) \cong B\,\mathbf{x}_o(t) + \xi_o(t) + CD\,\mathbf{x}_o(t-\Delta t) + CED\,\mathbf{x}_o(t-2\Delta t)\ .$$

$$\|E\|^k \cong 0 \text{ for } k > 1$$

Matrix reconstruction

$$\mathbf{x}_o(t+\Delta t) \cong [B, CD, CED]\,\mathbf{X}(t) + \xi_o(t)\,.$$

$$\mathbf{X}(t+2\Delta t) = [\mathbf{x}_o(t+2\Delta t), \mathbf{x}_o(t+\Delta t), \mathbf{x}_o(t)]^\top$$

Network reconstruction

Matrix reconstruction

$$\mathbf{x}_o(t+\Delta t) \cong [B, CD, CED]\,\mathbf{X}(t) + \xi_o(t)\,.$$

$$\Sigma_1 \cong [B, CD, CED]\,\Sigma_0\,.$$

$$\Sigma_0 = \frac{1}{N_t - 3}\sum_{k=2}^{N_t-2} \mathbf{X}(k\Delta t)\mathbf{X}^\top(k\Delta t)$$

$$\Sigma_1 = \frac{1}{N_t - 3}\sum_{k=2}^{N_t-2} \mathbf{x}_o((k+1)\Delta t)\mathbf{X}^\top(k\Delta t)\,.$$

Network reconstruction

Matrix reconstruction

$$\mathbf{x}_o(t+\Delta t) \cong [B, CD, CED]\,\mathbf{X}(t) + \xi_o(t)\,.$$

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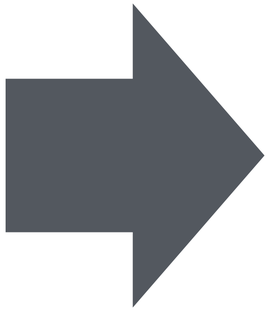


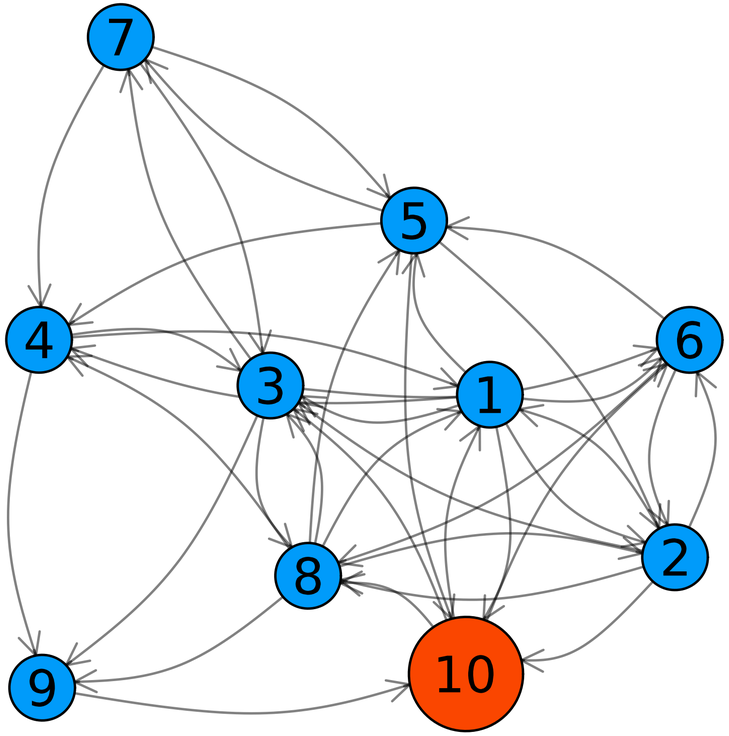
$$[\,\widehat{B}\,,\,\widehat{CD}\,,\,\widehat{CED}\,] = \Sigma_1\Sigma_0^{-1}\,.$$

Network reconstruction

Single hidden leader

$$B \in \mathbb{R}^{(N-1)\times(N-1)},$$
$$C \in \mathbb{R}^{N-1},$$
$$D \in \mathbb{R}^{N-1},$$
$$E \in \mathbb{R}.$$

$$[\widehat{B}, \widehat{CD}, \widehat{CED}] = \Sigma_1 \Sigma_0^{-1}.$$



$$\widehat{C}_i = 1 - \sum_{j=1}^{N_f} \widehat{B}_{ij},$$
$$\widehat{C}\,\widehat{D} = \widehat{CD},$$
$$\widehat{E} = M^{-1} \sum_{i,j \in \mathcal{M}} \frac{\widehat{CED}_{ij}}{\widehat{CD}_{ij}},$$
$$\widehat{\alpha_{N_l}} = \widehat{E} + \sum_{j=1}^{N_f} \widehat{D}_j.$$

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Single hidden leader

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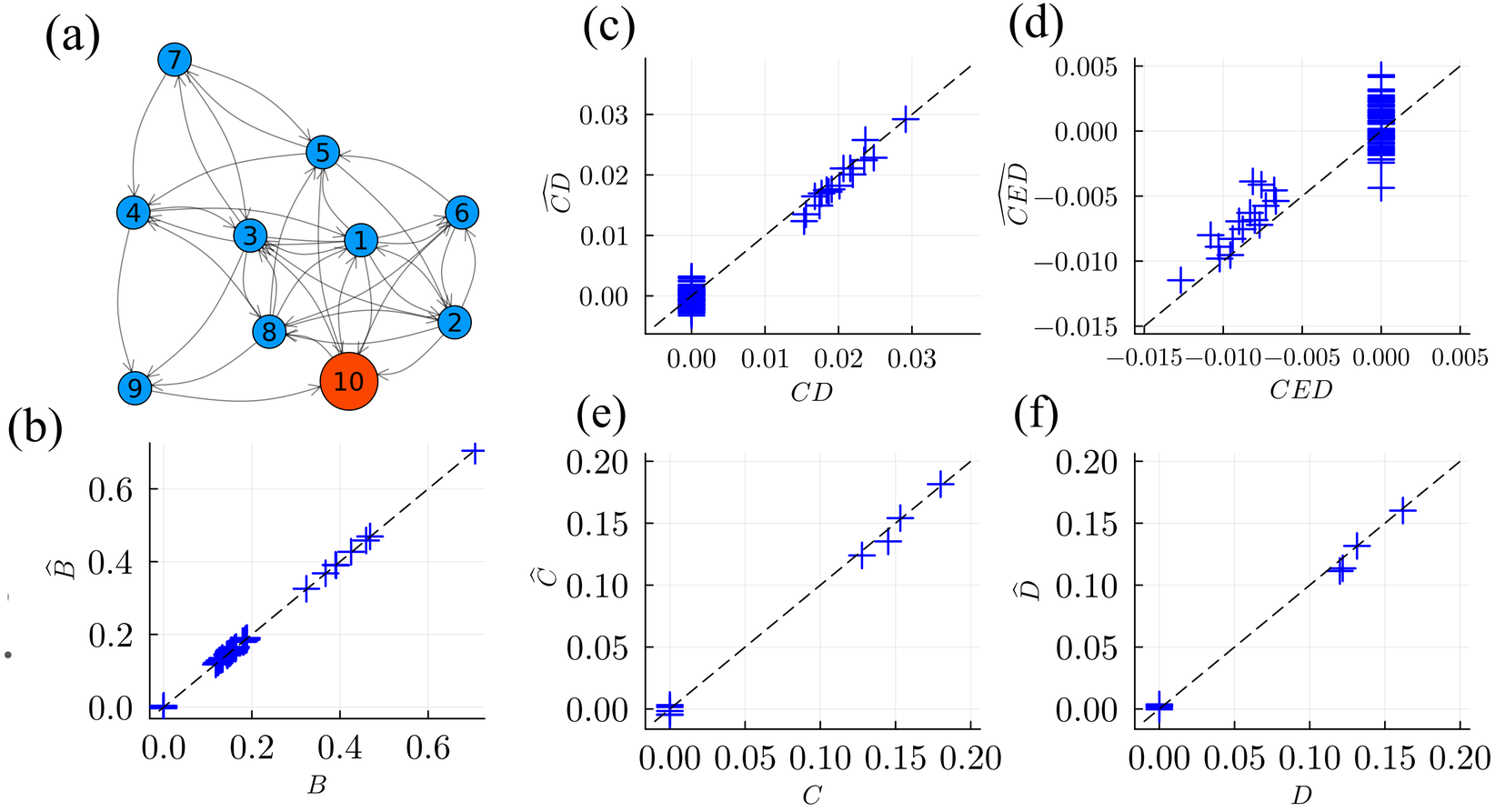
$$[\widehat{B}, \widehat{CD}, \widehat{CED}] = \Sigma_1 \Sigma_0^{-1}.$$



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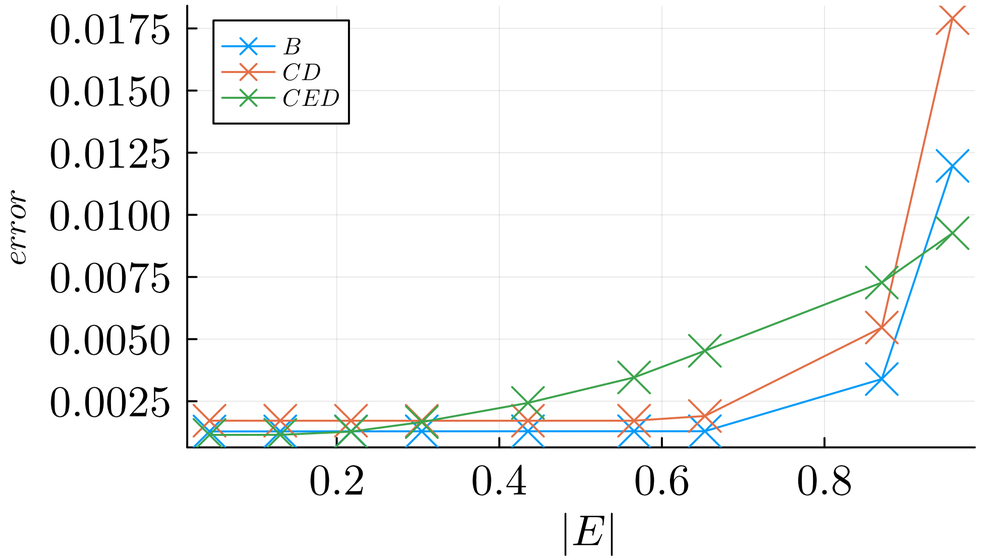
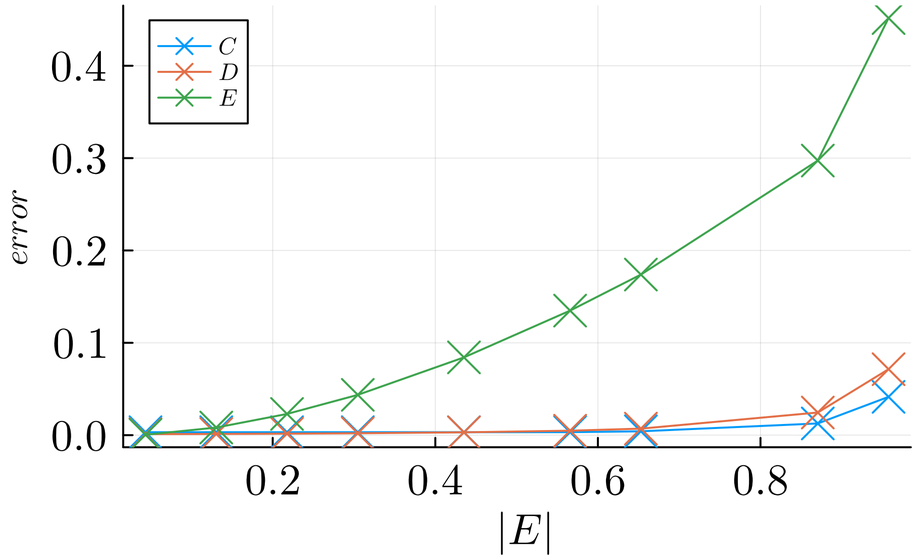
$$D \in \mathbb{R}^{N-1},$$
$$E \in \mathbb{R}.$$

$$[\widehat{B}, \widehat{CD}, \widehat{CED}] = \Sigma_1 \Sigma_0^{-1}.$$



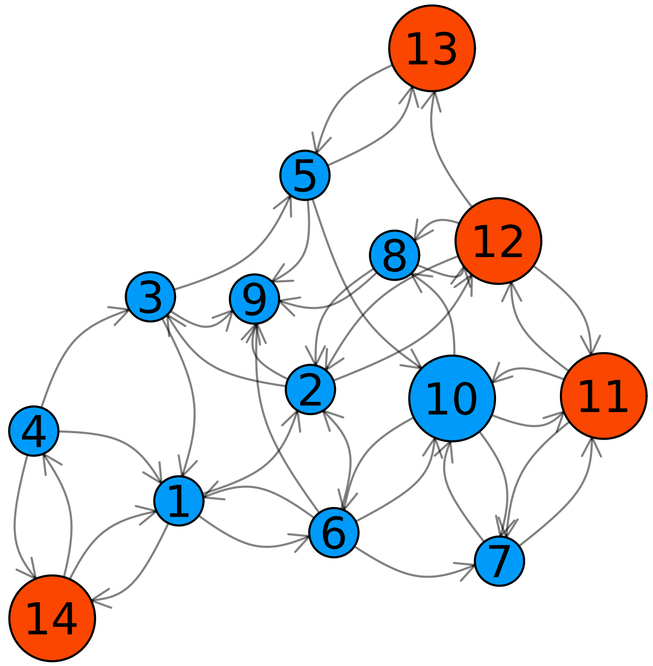


Network reconstruction

Multiple hidden leader

$$B \in \mathbb{R}^{(N-4)\times(N-4)},$$
$$C \in \mathbb{R}^{N-4},$$
$$D \in \mathbb{R}^{N-4},$$
$$E \in \mathbb{R}^{4\times 4}.$$

$$[\,\widehat{B}\,,\,\widehat{CD}\,,\,\widehat{CED}\,] = \Sigma_1 \Sigma_0^{-1}.$$



$$\widehat{C}_{:,i} = \frac{\widehat{CC^\top}_{:,j(i)}}{\left(\widehat{CC^\top}_{j(i)j(i)}\right)^{1/2}},$$

$$\widehat{C}\,\widehat{E}\,\widehat{C}^\top = \widehat{CEC^\top},$$

$$\widehat{\alpha_{N_f+i}} = \sum_{j=1}^{N_f} \widehat{C}_{ji} + \sum_{j=1}^{N_l} \widehat{E}_{ij}.$$

(i) Leaders symmetrically coupled to at least one follower each, i.e. $D = C^\top$

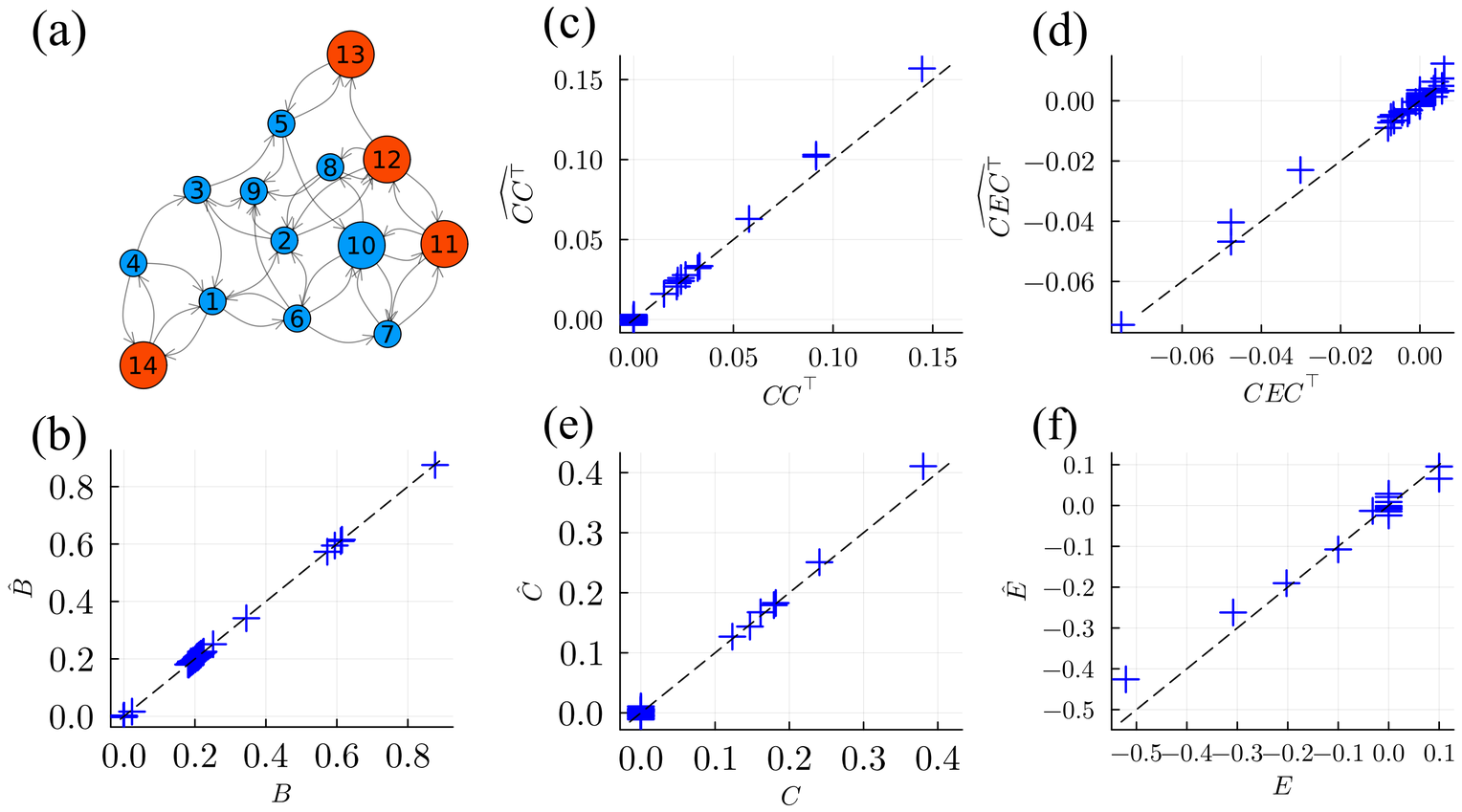
(ii) Leaders are not connected to the same followers.

Network reconstruction

Multiple hidden leader

$$B \in \mathbb{R}^{(N-4)\times(N-4)},$$
$$C \in \mathbb{R}^{N-4},$$
$$D \in \mathbb{R}^{N-4},$$
$$E \in \mathbb{R}^{4\times 4}.$$

$$[\widehat{B}, \widehat{CD}, \widehat{CED}] = \Sigma_1 \Sigma_0^{-1}.$$



Network reconstruction

Multiple hidden leader

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$$C \in \mathbb{R}^{N-4},$$
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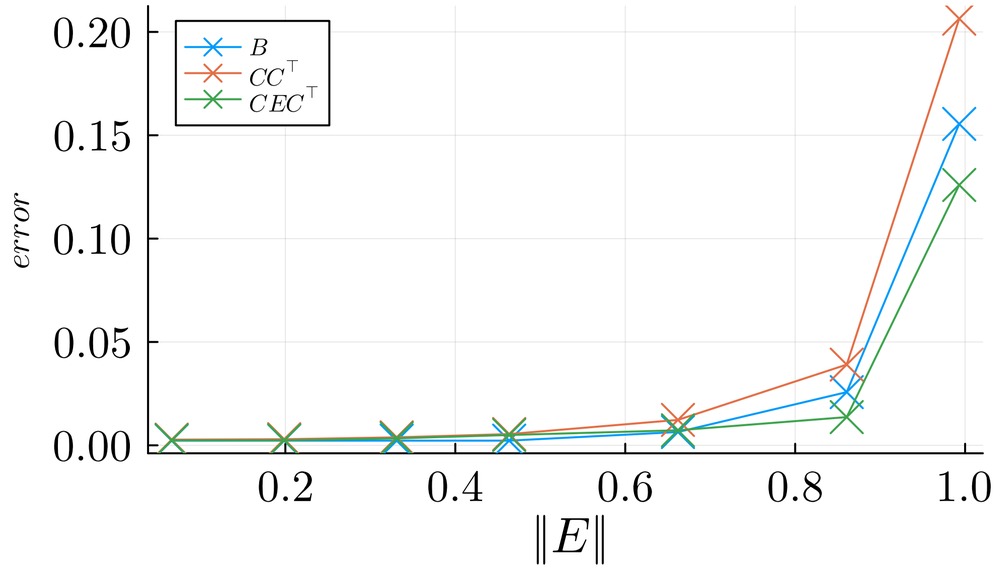
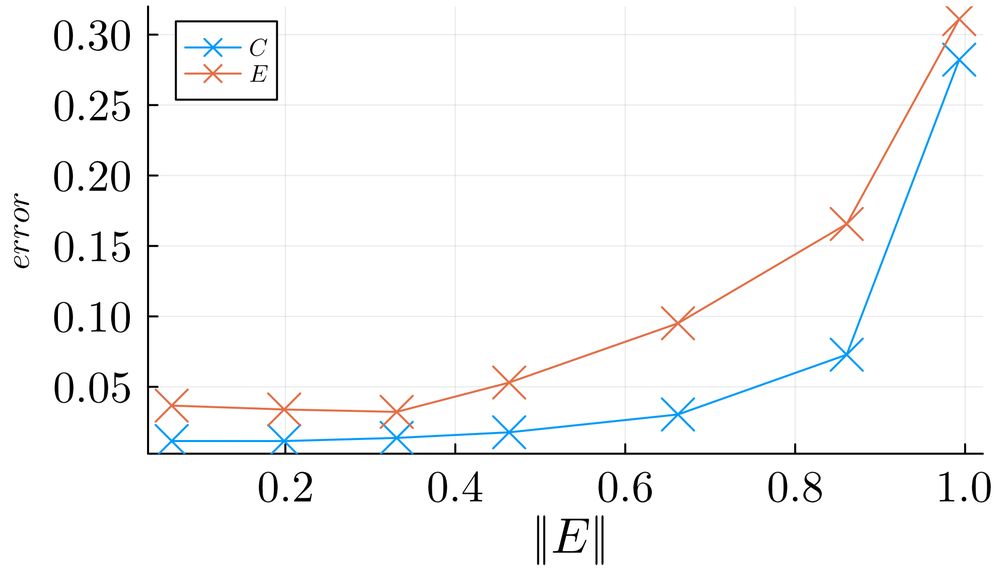
$$[\widehat{B}, \widehat{CD}, \widehat{CED}] = \Sigma_1 \Sigma_0^{-1}.$$





Conclusion: Part 2

- Model-based reconstruction of the full dynamical system.

Future work

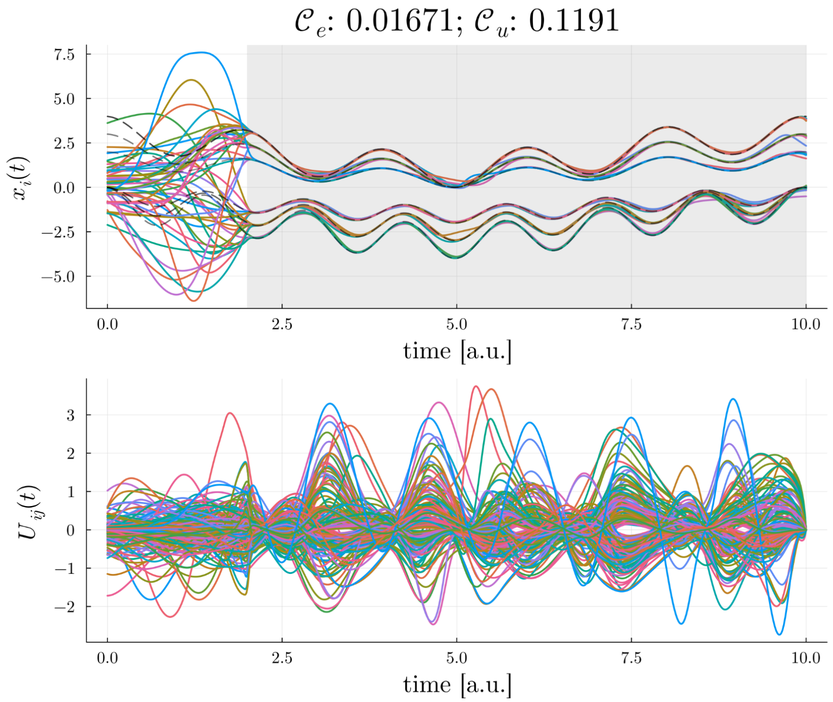
- Consider more terms in the expansion;
- Look at equivalent dynamical systems;
- Finite sample scaling;
- Actual data…

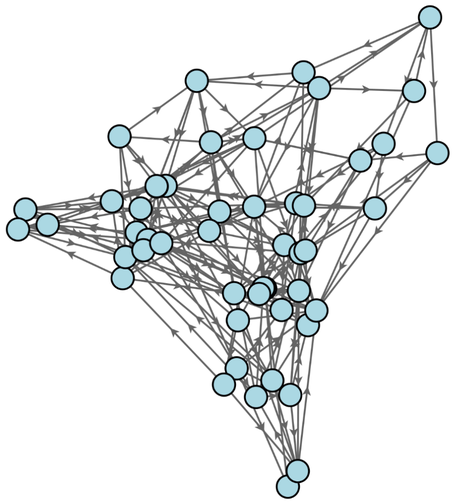
*MT, EPL **156**, 51001 (2026)*

Other stuff

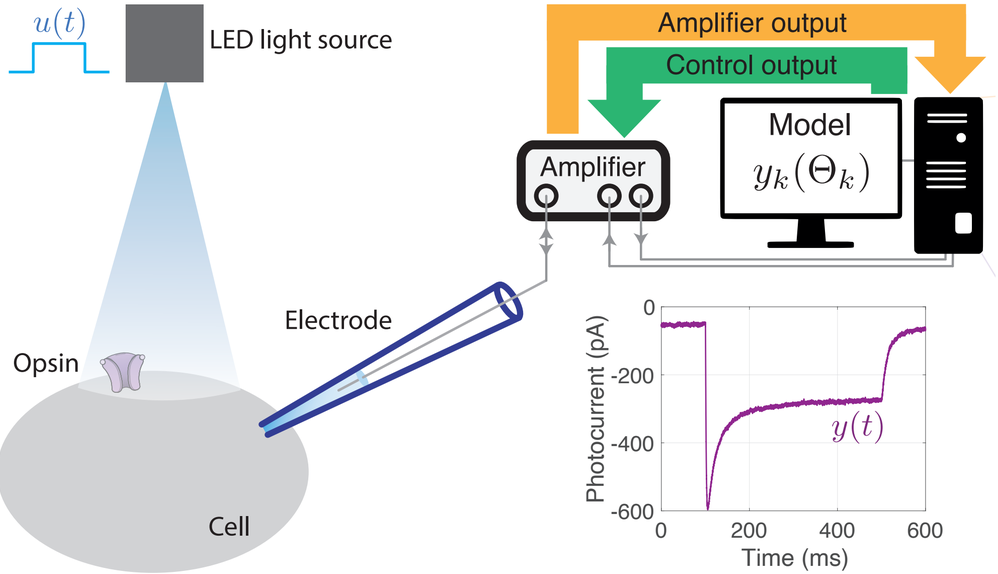
University of Exeter

Optimal control





Neuroscience



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