Robustness of Synchrony in Electrical Power Grids and Generalized Kirchhoff Indices

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We define measures to assess the excursion from the operational state of power grids due to perturbations. We relate these measures to new topological indices which we introduce as *generalized Kirchhoff indices*.

1: Swing Equations

The transient dynamics of a power grid is governed by the Swing equations

$$I_i\ddot{\theta}_i + D_i\dot{\theta}_i = P_i - \sum_j \Big(B_{ij}|V_i||V_j|\sin(\theta_i-\theta_j) + G_{ij}|V_i|[|V_i| - |V_j|\cos(\theta_i-\theta_j)]\Big)\,, \qquad (1)$$

- I_i, D_i the inertia and the damping at node i.
- $V_j = |V_i|e^{i\theta_j}$ complex voltage at node j.
- $P_i \geq 0\,(\leq 0)$ injected (consumed) active power at node i.
- B_{ij} and G_{ij} susceptance and conductance of the line connecting nodes i and j.

Working assumptions

- Consider PV-nodes and neglect voltage fluctuations $|V_i| = 1$.
- High voltage AC transmission lines have $G_{ij}/B_{ij} \approx 5\% - 10\%$.
  Neglecting the conductance (lossless approximation), the active power flow on a line is $P_{ij} = B_{ij}\sin(\theta_i - \theta_j)$.
  In this limit, analogy with the DC Josephson current between superconducting islands.

The considered model

$$\dot{\theta}_i = P_i - \sum_j B_{ij}\sin(\theta_i-\theta_j)\,. \qquad (2)$$

- Operational states of the power grid are stable fixed points of Eq. (2).
- We neglect the second order time derivative but the following calculations can be carried out with inertia.

2: Linearization around a stable fixed point

We consider the power grid at a stable fixed point $\{\theta_i^{(0)}\}$ for the injections $\{P_i^{(0)}\}$. Adding a perturbation in the injections such that $P_i(t) = P_i^{(0)} + \delta P(t)$, phases become time dependent $\theta_i(t) = \theta_i^{(0)} + \delta\theta_i(t)$ and we get the vectorial equation,

$$\delta\dot{\boldsymbol{\theta}} = \delta\boldsymbol{P} - \mathbb{L}(\{\theta_i^{(0)}\})\,\delta\boldsymbol{\theta}\,, \qquad (3)$$

where we introduced the weighted Laplacian matrix $\mathbb{L}(\{\theta_i^{(0)}\})$ with matrix elements

$$\mathbb{L}_{ij} = \begin{cases} -B_{ij}\cos(\theta_i^{(0)} - \theta_j^{(0)})\,, & i \neq j\,,\\ \sum_k B_{ik}\cos(\theta_i^{(0)} - \theta_k^{(0)})\,, & i = j\,. \end{cases} \qquad (4)$$

$\mathbb{L}(\{\theta_i^{(0)}\})$ has a single eigenvalue $\lambda_1 = 0$ with eigenvector $\mathbf{u}_1 = (1,1,1,...1)/\sqrt{n}$, and $\lambda_i > 0$, $i = 2,3,...n$. Eq. (3) can be solved by expanding the angle deviation over the eigenstate of $\mathbb{L}(\{\theta_i^{(0)}\})$,

$$\delta\boldsymbol{\theta}(t) = \sum_\alpha c_\alpha(t)\mathbf{u}_\alpha\,, \qquad (5)$$

with

$$c_\alpha(t) = e^{-\lambda_\alpha t}c_\alpha(0) + e^{-\lambda_\alpha t}\int_0^t \mathrm{d}t' e^{\lambda_\alpha t'}\,\delta\boldsymbol{P}(t')\cdot\mathbf{u}_\alpha\,. \qquad (6)$$

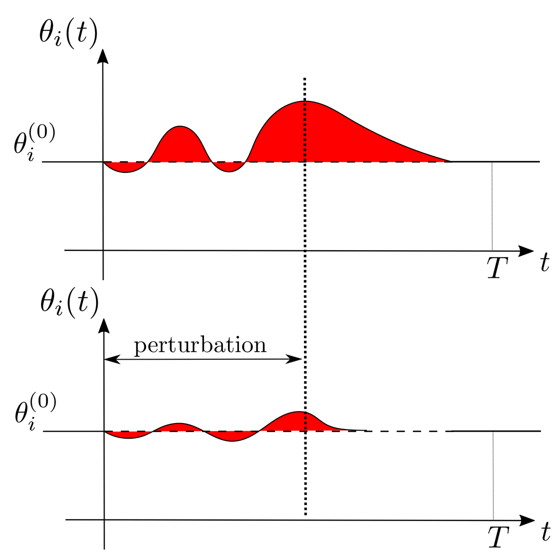
3: Fragility Measures

To assess the magnitude of this excursion in the spirit of Refs. [2, 3, 4] we consider two fragility performance measures

$$\mathcal{C}_1(T) = \sum_i \int_0^T |\delta\theta_i(t) - \Delta(t)|^2\mathrm{d}t = \sum_{\alpha\geq 2}\int_0^T c_\alpha^2(t)\mathrm{d}t\,, \qquad (7a)$$

$$\mathcal{C}_2(T) = \sum_i \int_0^T |\delta\dot{\theta}_i(t) - \dot{\Delta}(t)|^2\mathrm{d}t = \sum_{\alpha\geq 2}\int_0^T \dot{c}_\alpha^2(t)\mathrm{d}t\,. \qquad (7b)$$

with $\Delta(t) = n^{-1}\sum_j \delta\theta_j(t)$ and $\dot{\Delta}(t) = n^{-1}\sum_j \delta\dot{\theta}_j(t)$. In power grids $\mathcal{C}_1(T)$ is known as the coherence of the operational state and $\mathcal{C}_2(T)$ is proportional to the primary control effort.



4: Generalized Kirchhoff Indices

The Kirchhoff index originally followed from the definition of the resistance distance in a graph [5]. To a connected graph, one associates an electrical network where each edge is a resistor given by the inverse edge weight in the original graph. The resistance distance is the resistance Ω_{ij} between any two nodes i and j on the electrical network. The Kirchhoff index is then defined as [5]

$$Kf_1 \equiv \sum_{i<j}\Omega_{ij}\,, \qquad (8)$$

where the sum runs over all pairs of nodes in the graph. For a graph with Laplacian $\mathbb{L}$, it has been shown that Kf_1 is given by the spectrum $\{\lambda_\alpha\}$ of $\mathbb{L}$ as [6, 7]

$$Kf_1 = n\sum_{\alpha\geq 2}\lambda_\alpha^{-1}\,. \qquad (9)$$

Up to a normalization prefactor, Kf_1 gives the mean resistance distance $\overline{\Omega}$ over the whole graph. Intuitively, one expects the dynamics of a complex system to depend not only on $\overline{\Omega}$, but on the full set $\{\Omega_{ij}\}$. Higher moments of $\{\Omega_{ij}\}$ are encoded in generalized Kirchhoff indices Kf_m which we define as

$$Kf_m = n\sum_{\alpha\geq 2}\lambda_\alpha^{-m}\,, \qquad (10)$$

for integers m. Below we show that $\mathcal{C}_{1,2}$ can be expressed as linear combinations of the Kf_m's corresponding to $\mathbb{L}$ in Eq. (4).

5: Fragility Measures as functions of Kf_m's

Fragility measures $\mathcal{C}_1$ and $\mathcal{C}_2$ can be computed for various type of $\delta\boldsymbol{P}(t)$ [1]. Here we show results for a time correlated noisy perturbation. On the right we compare our analytical results to numerics for a cyclic network with nearest and q -th neighbor coupling (see Ref. [1] for more numerics and other types of perturbations) :

- **Noisy perturbation** : $\overline{\delta P_i(t_1)\,\delta P_j(t_2)} = \delta_{ij}\delta P_{0i}^2\exp[-|t_1-t_2|/\tau_0]$

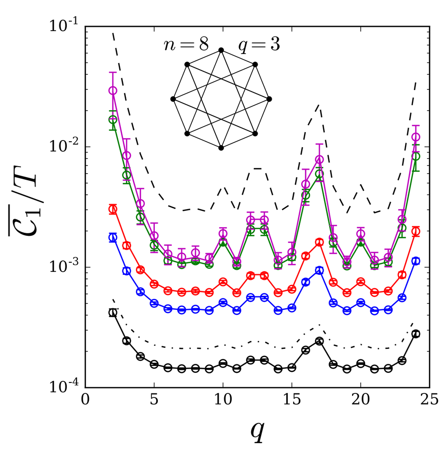
$$\overline{\mathcal{C}_1}(T) = T\sum_\alpha \frac{\sum_{i\in N_{\rm n}}\delta P_{0i}^2\,u_{\alpha,i}^2}{\lambda_\alpha(\lambda_\alpha + \tau_0^{-1})} + \mathcal{O}(T^0)\,, \qquad (11a)$$

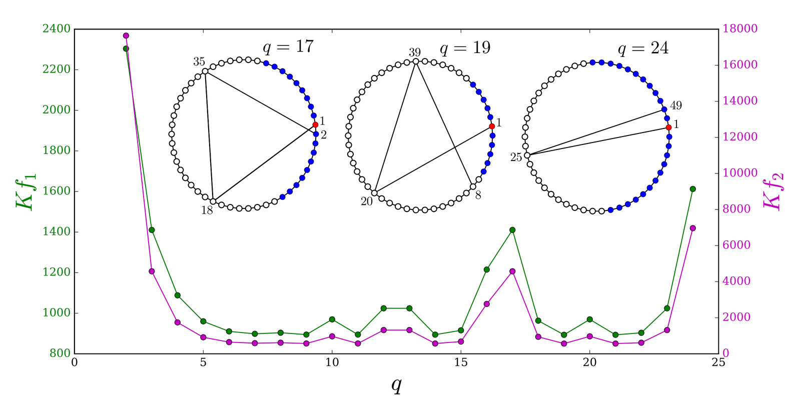
$$\overline{\mathcal{C}_2}(T) = (T/\tau_0)\sum_\alpha \frac{\sum_{i\in N_{\rm n}}\delta P_{0i}^2\,u_{\alpha,i}^2}{\lambda_\alpha + \tau_0^{-1}} + \mathcal{O}(T^0)\,. \qquad (11b)$$

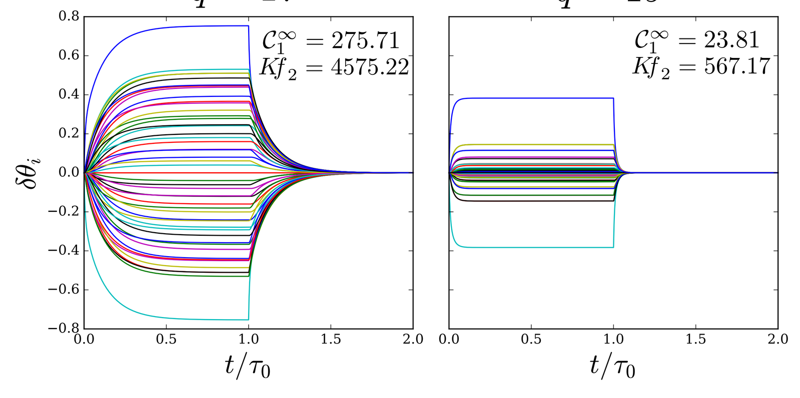
Averaging over an ensemble of perturbations and if τ_0^{-1} lies outside the spectrum of $\mathbb{L}$, the measures are directly expressable as infinite sums over Kf_m's, $\langle\mathcal{C}_{1,2}\rangle = n^{-1}\langle\delta P_0^2\rangle\, T\sum_{m=0}^\infty C_{1,2}^{(m)}$ with

$$C_1^{(m)} = \begin{cases} (-1)^m\,\tau_0^{(m+1)}Kf_{-m+1}\,, & \lambda_\alpha\tau_0 < 1\,,\\ (-1)^m\,\tau_0^{-m}Kf_{m+2}\,, & \lambda_\alpha\tau_0 > 1\,, \end{cases} \qquad (12a)$$

$$C_2^{(m)} = \begin{cases} (-1)^m\,\tau_0^{m}Kf_{-m}\,, & \lambda_\alpha\tau_0 < 1\,,\\ (-1)^m\,\tau_0^{-(m+1)}Kf_{m+1}\,, & \lambda_\alpha\tau_0 > 1\,. \end{cases} \qquad (12b)$$







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6: Conclusion

We have shown that the performance measures $\mathcal{C}_1$ and $\mathcal{C}_2$ depend on the overlap between the perturbation vector $\delta\boldsymbol{P}_0$ and the eigenmodes $\mathbf{u}_\alpha$ of the weighted Laplacian matrix $\mathbb{L}(\{\theta_i^{(0)}\})$. From Eqs. (11)-(13) perturbations along the eigenmodes with smallest Lyapunov exponents have the largest impact on the synchronous state. After averaging over an ensemble of perturbation vectors, these measures can be expressed as functions of new indices which we introduced as generalized Kirchhoff indices and depend on both the topology of the network and the stable fixed point of the dynamical system.

- Kf_m's → network's global/average fragility.
- Kf_m's → easy to compute; only have to determine few of the smallest λ_α.

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