Evolution of Robustness in Growing Random Networks

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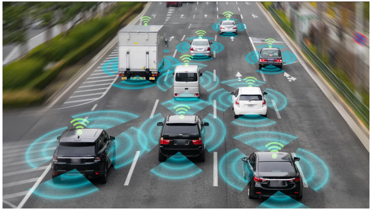
MT, Entropy **25**, 1340 (2023).

Growings Coupled Dynamical Systems

Autonomous vehicular platoon



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Growings Coupled Dynamical Systems

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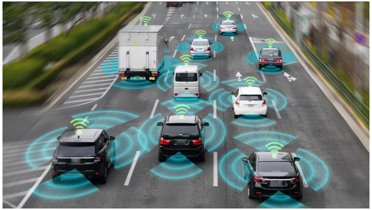

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Electric power grids

Growings Coupled Dynamical Systems

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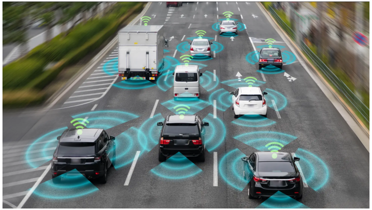

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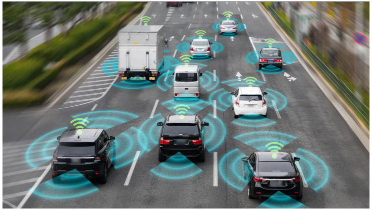
Electric power grids

Collective states

Growings Coupled Dynamical Systems

Autonomous vehicular platoon



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Electric power grids

Collective states →Evolution of robustness

Network realization

Optimization in complex networks

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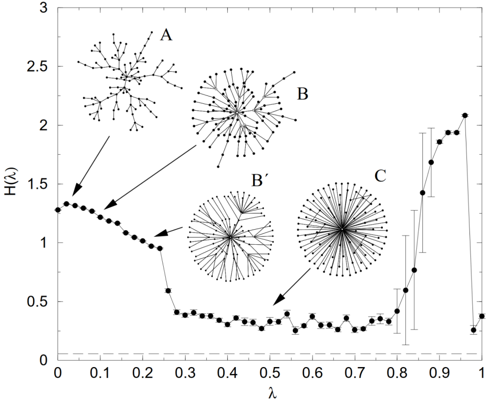
[3] *Santa Fe Institute, 1399 Hyde Park Road, New Mexico 87501, USA.*

(Dated: October 15, 2018)



Growings Networks

VOLUME 85, NUMBER 21 PHYSICAL REVIEW LETTERS 20 NOVEMBER 2000

Connectivity of Growing Random Networks

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Structure of Growing Networks with Preferential Linking

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Diffusively coupled systems

Linear coupled oscillators

$$\dot{x}_i = -\sum_{j=1}^{N} a_{ij}(x_i - x_j) + \eta_i \,, \quad i = 1, ...N \,, \tag{1}$$

Adjacency matrix elements: $a_{ij} = a_{ji} > 0$
White-noise inputs: $\langle \eta_i(t)\eta_j(t')\rangle = \eta_0^2 \, \delta_{ij} \, \delta(t - t)$.

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$$\frac{1}{N}\sum_{j=1}^{N}\langle x_j^2\rangle = \frac{\eta_0^2}{2}Kf_1/N\,, \tag{2}$$

Network descriptors

Kirchhoff Index

$$Kf_1 = \sum_{i<j} \Omega_{ij} \,. \tag{3}$$

The resistance distance between node i and j is defined by,

$$\Omega_{ij} = [\mathbb{L}^\dagger]_{ii} - 2[\mathbb{L}^\dagger]_{ij} + [\mathbb{L}^\dagger]_{jj} \,, \tag{4}$$

$$\mathbb{L}_{ij} = \begin{cases} a_{ij} & i \neq j \\ -\sum_{k=1}^{N} a_{ik} & i = j \,, \end{cases} \tag{5}$$

D. J. Klein and M. Randić , Journal of Mathematical Chemistry **12**, 81-95 (1993).

Kirchhoff Index

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$$Kf_1 = N \sum_{\alpha>1} \frac{1}{\lambda_\alpha} = N\,\mathrm{Tr}[\mathbb{L}^\dagger]\,. \tag{6}$$

D. J. Klein and M. Randić , Journal of Mathematical Chemistry **12**, 81-95 (1993).

Growing network

Questions

1. How does Kf_1 evolves when the network is growing?
2. Can we control it?

Adding an edge

$$Kf_k(t+1) = Kf_1(t) - \frac{a_{kl}\mathrm{Tr}[\mathbb{L}^\dagger \boldsymbol{e}_{kl}\boldsymbol{e}_{kl}^\top \mathbb{L}^\dagger]}{1 + a_{kl}\Omega_{lk}(t)} = Kf_1(t) - N_t\left[\frac{a_{kl}\Omega_{kl}^{(2)}(t)}{1 + a_{kl}\Omega_{kl}(t)}\right], \quad (7)$$

where $\Omega_{kl}^{(2)}(t) = \sum_{\alpha>1}(u_{\alpha,i} - u_{\alpha,j})^2/\lambda_\alpha^2$ is a semi-metric.

Robustness of growing network

Adding an edge

$$Kf_k(t+1) = Kf_1(t) - \frac{a_{kl}\mathrm{Tr}[\mathbb{L}^\dagger \boldsymbol{e}_{kl}\boldsymbol{e}_{kl}^\top \mathbb{L}^\dagger]}{1 + a_{kl}\Omega_{lk}(t)} = Kf_1(t) - N_t\left[\frac{a_{kl}\Omega_{kl}^{(2)}(t)}{1 + a_{kl}\Omega_{kl}(t)}\right], \qquad (7)$$

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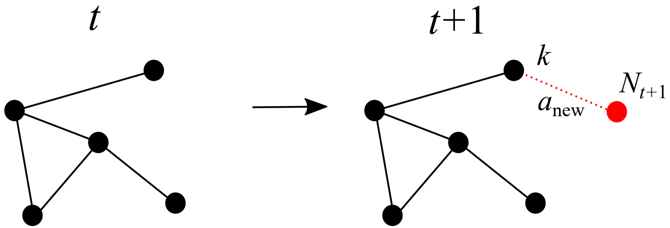
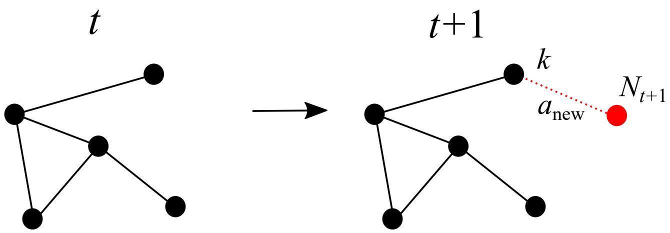
Only reduces the Kirchhoff index.

Adding one node with one edge



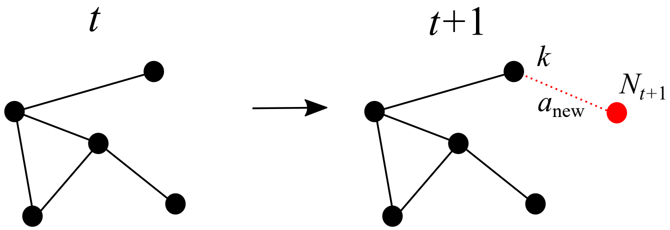
Robustness of growing network

Adding one node with one edge



$$Kf_1(t+1) = Kf_1(t) + \sum_{l=1}^{N_t} \Omega_{kl}(t) + \frac{N_t}{a_{\rm new}}\,, \qquad (8)$$

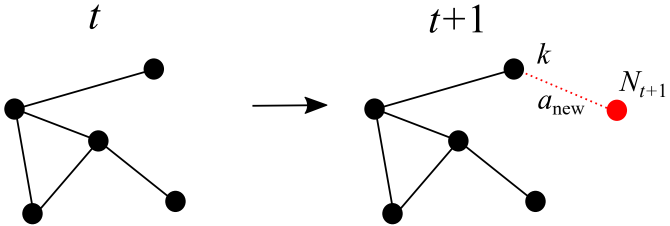
Adding one node with one edge



Worst case

$$\sum_{l=1}^{N_t} \Omega_{kl}(t) \cong \frac{N_t(N_t-1)}{2}\,. \tag{9}$$

Adding one node with one edge



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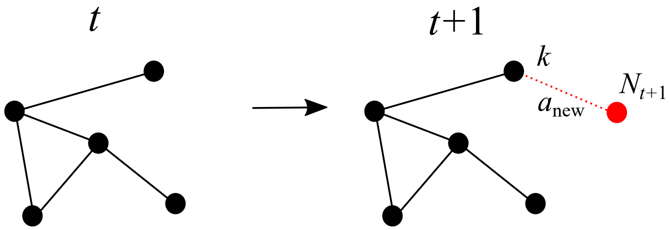
$$Kf_1(t+1) \;\cong\; Kf_1(t) + \frac{N_t(N_t-1)}{2} + \frac{N_t}{a_{\text{new}}} \overset{t\to\infty}{\propto} N_t^3\,, \tag{10}$$

Adding one node with one edge



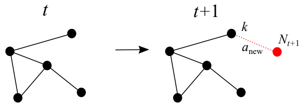
Best case

$$\sum_{l=1}^{N_t} \Omega_{kl}(t) \cong (N_t - 1)\,, \tag{11}$$

$$Kf_1(t+1) \cong Kf_1(t) + (N_t - 1) + \frac{N_t}{a_{\rm new}} \overset{t\to\infty}{\propto} N_t^2\,. \tag{12}$$

Robustness of growing network

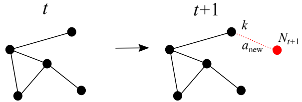
Adding one node with one edge



Random case

$$\langle Kf_1(t+1)\rangle = \langle Kf_1(t)\rangle \left(1+\frac{2}{N_t}\right) + \frac{N_t}{a_{\rm new}} \tag{13}$$

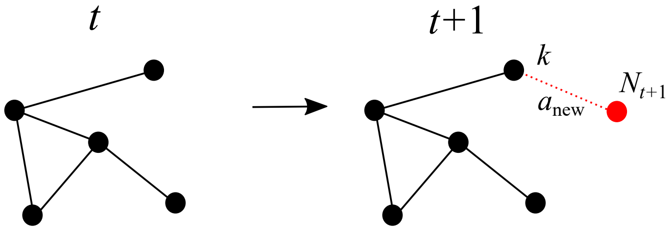
Adding one node with one edge



Random case

$$\begin{aligned}
\langle Kf_1(t+1)\rangle &= \langle Kf_1(t)\rangle\left(1+\frac{2}{N_t}\right)+\frac{N_t}{a_{\rm new}} && (13)\\
&= \frac{(N_0+t+1)}{a_{\rm new}}\left[\frac{a_{\rm new}\frac{Kf_1(0)(N_0+2)}{N_0}(N_0+t+2)-2(N_0+2)(t+1)}{(N_0+1)(N_0+2)}\right.\\
&+ \left.(N_0+t+2)(H_{N_0+t+1}-H_{N_0})\right] && (14)\\
&= \frac{(N_0+t+1)}{a_{\rm new}}\left[\frac{a_{\rm new}\frac{Kf_1(0)(N_0+2)}{N_0}(N_0+t+2)-2(N_0+2)(t+1)}{(N_0+1)(N_0+2)}\right.\\
&+ \left.(N_0+t+2)\{\psi_0(N_0+t+1)-\psi_0(N_0)\}\right]\,, && (15)
\end{aligned}$$

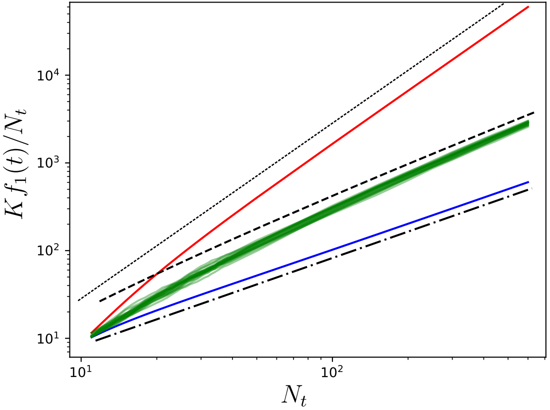
Adding one node with one edge



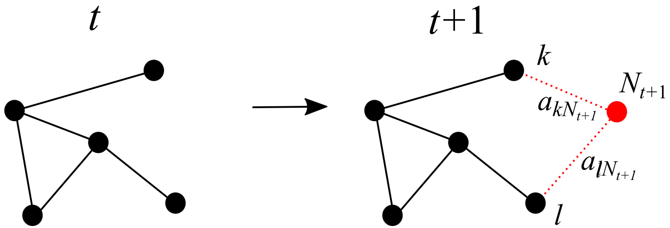
Random case

$$\langle Kf_1(t)\rangle \overset{t\to\infty}{\propto} N_t^2 \log N_t \,. \tag{16}$$

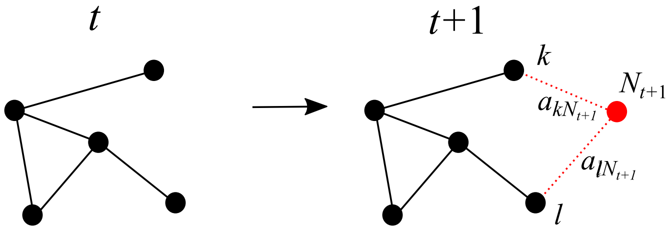
Adding one node with one edge



Adding one node with two edges



Adding one node with two edges



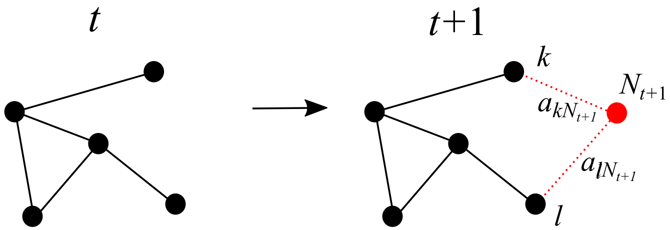
$$\omega_{kl} = a_{k\,N_{t+1}}^{-1} + a_{l\,N_{t+1}}^{-1} \,. \tag{17}$$

$$Kf_k(t+1) = \frac{1}{2}\sum_{i,j=1}^{N_t}\Omega_{ij}(t+1) + \sum_{i=1}^{N_t}\Omega_{iN_{t+1}}(t+1) \tag{18}$$

$$= Kf_1(t) - N_t\left[\frac{\omega_{kl}^{-1}\,\Omega_{kl}^{(2)}(t)}{1+\omega_{kl}^{-1}\Omega_{lk}(t)}\right] + \sum_{i=1}^{N_t}\Omega_{iN_{t+1}}(t+1)\,. \tag{19}$$

Robustness of growing network

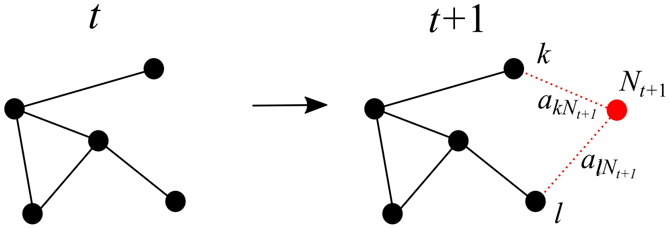
Adding one node with two edges



$$\omega_{kl} = a_{k\,N_{t+1}}^{-1} + a_{l\,N_{t+1}}^{-1}\,. \tag{20}$$

$$\Omega_{ij}(t+1) = \Omega_{ij}(t) - \frac{\omega_{kl}^{-1}\,[\mathbf{e}_{ij}^{\top}\mathbb{L}^{\dagger}(t)\mathbf{e}_{kl}]^2}{1+\omega_{kl}^{-1}\Omega_{kl}(t)}\,,\quad i,j=1,...,N_t\,, \tag{21}$$

Adding one node with two edges



$$\omega_{kl} = a^{-1}_{k\,N_{t+1}} + a^{-1}_{l\,N_{t+1}}\,. \tag{22}$$

$$\sum_{i=1}^{N_t} \Omega_{iN_{t+1}}(t+1) \cong \frac{1}{\left(a_{k\,N_{t+1}} + a_{l\,N_{t+1}}\right)} \sum_{j=1}^{N_t} \left[a_{kj}\Omega_{kj}(t+1) + a_{lj}\Omega_{lj}(t+1)\right]. \tag{23}$$

Adding one node with two edges

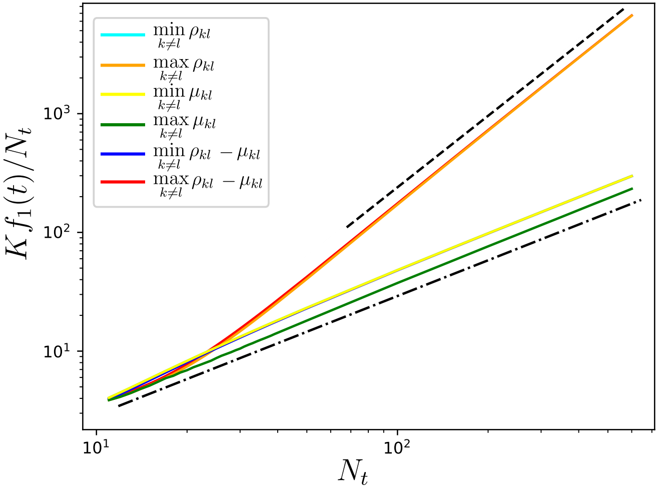
$$Kf_1(t+1) = \rho_{kl}(t) - \mu_{kl}(t) \tag{24}$$

$$\mu_{kl}(t) = \frac{\omega_{kl}^{-1}}{1+\omega_{kl}^{-1}\Omega_{lk}(t)}\Bigg\{(N_t+1)\Omega_{kl}^{(2)}(t) + N_t\frac{\Omega_{kl}^2(t)}{2} + N_t\frac{\left[C^{-1}(k,t)-C^{-1}(l,t)\right]^2}{2} + N_t\frac{(a_{k\,N_{t+1}} - a_{l\,N_{t+1}})}{(a_{k\,N_{t+1}} + a_{l\,N_{t+1}})}\left[C^{-1}(k,t)-C^{-1}(l,t)\right]\Omega_{kl}(t)\Bigg\}\,, \tag{25}$$

$$\rho_{kl}(t) = N_t\frac{a_{k\,N_{t+1}}C^{-1}(k,t) + a_{l\,N_{t+1}}C^{-1}(l,t)}{(a_{k\,N_{t+1}} + a_{l\,N_{t+1}})}\,. \tag{26}$$

Robustness of growing network

Adding one node with two edges



Robustness of growing network

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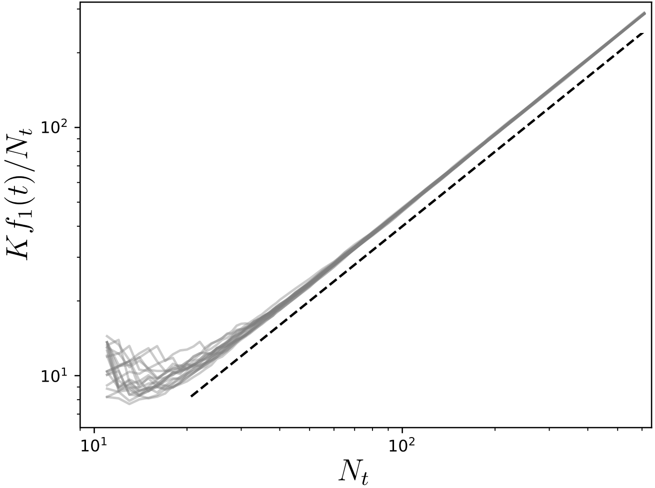

Conclusion and future work

So far

- Add edges $\rightarrow$ reduces the Kirchhoff index.
- Add nodes together with edges $\rightarrow$ different scalings achievable.
- Connection between the evolution of the Kirchhoff index and the nodes properties.

MT, Entropy **25**, 1340 (2023).

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So far

- Add edges $\rightarrow$ reduces the Kirchhoff index.
- Add nodes together with edges $\rightarrow$ different scalings achievable.
- Connection between the evolution of the Kirchhoff index and the nodes properties.

Future work

- Check other properties of the networks generated.
- Combination of growing mechanisms.

MT, Entropy **25**, 1340 (2023).