Robustness of synchronous networks

Melvyn Tyloo

Director's Postdoc Fellow, T-4 and CNLS

Los Alamos NATIONAL LABORATORY

website: melvyntyloo.com

mtyloo@lanl.gov — 4/16/23

Synchronization and complexity

Check for updates | editorial

Complexity matters

This month, we celebrate the fiftieth anniversary of Philip Anderson's landmark essay 'More is Different'.

If one asked physicists to describe Philip Anderson's influence in a single word, the most likely answers would be along the lines of 'broad' or 'wide'. Among condensed matter physicists in particular, Anderson is a legendary figure. His contributions have reshaped the understanding of interference phenomena in disordered media, magnetism, and of many other properties of quantum systems. In the early 1960s, he proposed a symmetry-breaking mechanism to explain how a photon could acquire mass within a superconductor, which was crucial for the subsequent development of the Higgs mechanism. But perhaps the truest testament of the far-reaching influence of his ideas is his 1972 essay 'More is Different'[1], which helped establish some of the philosophical foundations of complexity science.

Today, condensed matter physics is huge. But things were different — or at least perceived differently — in the 1970s when Anderson wrote the article. In his own words, More is Different "was unquestionably the result of a build-up of resentment and discontent on my part and among the condensed matter physicists[2]", mainly directed at colleagues that were part of "the particle physics establishment"[2]. What Anderson called the arrogance of particle physicists is well summarized by the distinction put forward by Victor Weisskopf, an eminent theoretical physicist, who was at the time director-general of CERN.

Weisskopf identified two major types of scientific research, one 'intensive', which "goes for the fundamental laws", and the other 'extensive', which explains phenomena "in terms of known fundamental laws"[3]. This might seem a reasonable, if coarse, categorization, but what bothered Anderson was the hierarchical structure that one infers from it. What Weisskopf implied — in Anderson's eyes — was that unveiling the microscopic laws that govern the behaviour of particles has an intrinsic priority and is the purest intellectual challenge. All else is an application of those laws, and condensed matter physics is nothing more than "Schmutzphysik" — the physics of dirt, as Wolfgang Pauli once said.

More is Different was the result of Anderson's urge to provide more dignity to his own research field. But it went far beyond that by dissecting the limitations of the philosophical approach that underpinned the perceived privilege of 'intensive' research fields. In his essay, Anderson borrowed and generalized the concept of emergence from evolutionary biology, laying out the idea that systems at a given scale have properties that cannot be directly predicted from the laws describing the behaviour of constituents at a lower scale. For example, consciousness is an emergent property of the brain, but neurons are not individually conscious. Similarly, knowing the inner workings of every single component of a car does not help us describe the complex patterns arising in traffic flows.

Anderson did not mean to fully disrupt the value of the opposite approach — reductionism. There is no doubt that the brain is made of neurons, and that the laws regulating the behaviour of microscopic entities hold in any context. But the point of More is Different is that a reductionist approach does not imply a 'constructionist' one. One cannot use the laws learned at a certain scale as building blocks to directly explain the emergent properties at higher scales. Reality is a collection of layers of emergence, and all the laws and frameworks needed to understand them share the same universal and fundamental quality. From this perspective, chemistry is not just applied physics, biology is not just applied chemistry — all the way up to sociology, which is not just applied psychology.

These ideas helped set a new direction in the study of complex phenomena as a deviation from the reductionist paradigm. Emergence is now considered one of the hallmarks of complex systems, in which the properties of the whole cannot be directly inferred from the details of the parts but arise from their mutual interactions. In this context, More is Different is not only a catchy slogan that crystallizes the concept of emergence, but it has been a crucial reading for generations of scientists willing to explore complex phenomena arising in widely different systems. Cities, neural networks, ecosystems and social media, are just a few examples of the rich variety that complexity science explores.

Despite the lasting impact that Anderson's call for a constructionist framework had in the physics community and beyond, seeds of reductionism are still embedded in the way we think. Ask any student who recently graduated from high school how to visualize the relationships among different scientific fields, and there is a good chance you will get a tree or pyramid diagram of some sort that ranks the disciplines by how 'fundamental' they are — social sciences at the top, mathematics and physics at the bottom. And ask any freshly enrolled physics student the reason behind their choice, and someone will almost certainly express the feeling that physics is more important or fundamental than other disciplines.

Fifty years ago, Anderson captured the need to shift gears. Thinking — and teaching — of science as a set of individual nodes, rigidly encapsulated in a vertical structure, contradicts the view of science as a complex system itself[4]. An adaptive network of people, ideas, theories and projects that communicate with each other, moulding the cultural landscape that we inhabit. More is Different is still an eye-opening read for those who have not yet realized how science has grown to be deeply interconnected. But it is also a reassuring experience for those who are not involved in 'intensive' research. Were physicists to ever achieve a theory of everything, it may have a large impact on fundamental fields. But the rest of physics and science as a whole would be far from done, as there would still be plenty of layers of complexity left to explore. ❐

Published online: 8 August 2022
https://doi.org/10.1038/s41567-022-01734-5

References

1. Anderson, P. W. *Science* **177**, 393–396 (1972).
2. Steglich, F. et al. *More is Different—Fifty Years of Condensed Matter Physics* (Princeton Univ. Press, 2001).
3. Weisskopf, V. F. *Brookhaven Nat. Lab. Publ.* 24–27 https://nemenmanlab.org/~ilya/images/a/a4/Weisskopf-1965.pdf (1965).
4. Fortunato, S. et al. *Science* **359** (2018).

4 August 1972, Volume 177, Number 4047 SCIENCE

More Is Different

Broken symmetry and the nature of the hierarchical structure of science.

P. W. Anderson

less relevance they seem to have to the very real problems of the rest of science, much less to those of society.

The constructionist hypothesis breaks down when confronted with the twin difficulties of scale and complexity. The behavior of large and complex aggregates of elementary particles, it turns out, is not to be understood in terms of a simple extrapolation of the properties of a few particles. Instead, at each level of complexity entirely new properties appear, and the understanding of the new behaviors requires re-

Editorial, Nature Physics **18**, 843 (2022)
Anderson, Science **177**(4047), 393-396 (1972)

Synchronization

- Simple experiment: coupled metronomes

Synchronization

- Simple experiment: coupled metronomes
- Other examples:
  - Fireflies flashing in unison
  - People clapping their hands or walking on a bridge
  - Synchronization of the quantum phase of Josephson junctions arrays
  - Synchronization of the phase of the voltages in electric power grids
  - Consensus formation on influence networks
  - Vehicular platoon formation
  - Orchestra

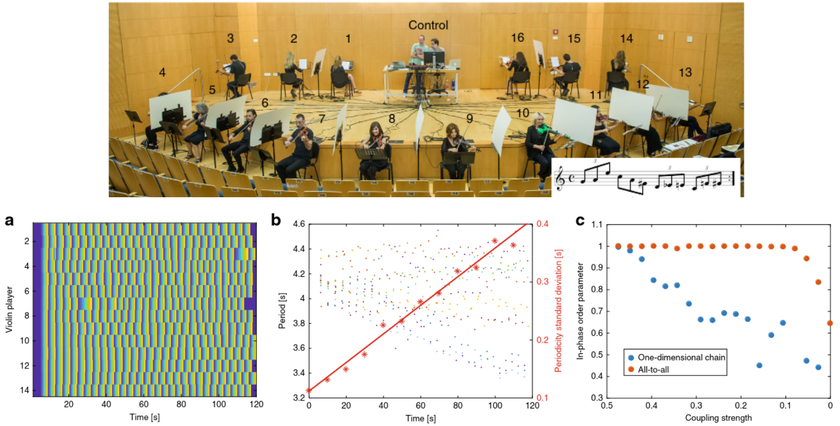
Synchronization and Music

NATURE COMMUNICATIONS | https://doi.org/10.1038/s41467-020-17540-7 ARTICLE

Shahal et al. Nature communications **11**, 3854 (2020).

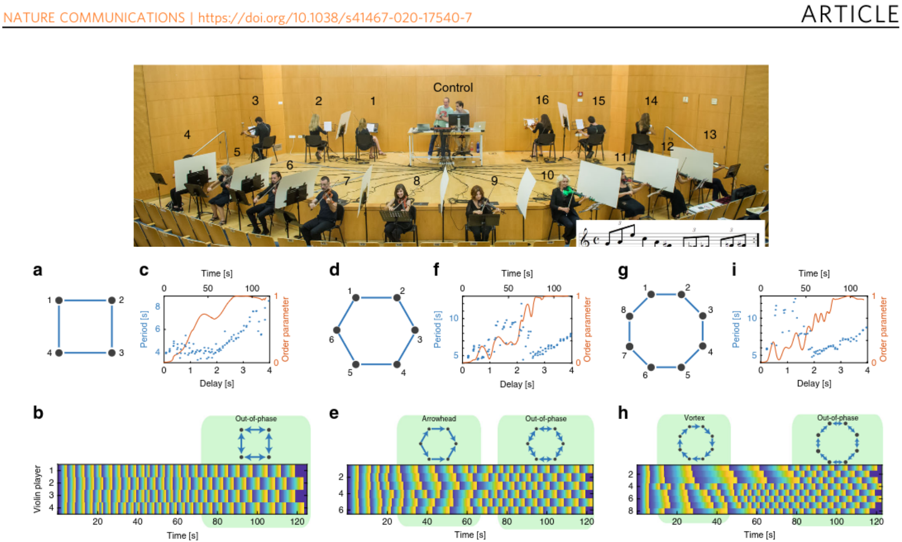
Synchronization and Music

NATURE COMMUNICATIONS | https://doi.org/10.1038/s41467-020-17540-7 ARTICLE



Shahal et al. Nature communications **11**, 3854 (2020).

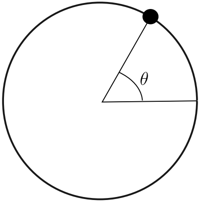
Synchronization and Music



Shahal et al. Nature communications **11**, 3854 (2020).

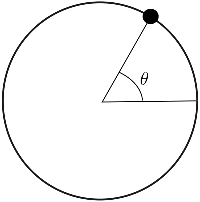
Modelling Synchronization

Single phase oscillator: $\dot{\theta} = \omega$

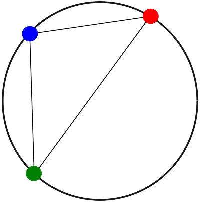


Modelling Synchronization

Single phase oscillator: $\dot{\theta} = \omega$

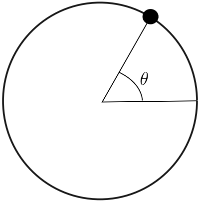


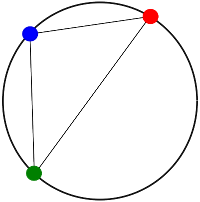
Coupled phase oscillators: $\dot{\theta}_i = \omega_i - \sum_j b_{ij} f(\theta_i - \theta_j)$



Modelling Synchronization

Single phase oscillator: $\dot{\theta} = \omega$





Coupled phase oscillators: $\dot{\theta}_i = \omega_i - \sum_j b_{ij} f(\theta_i - \theta_j)$

Synchonization: phase-locked $\dot{\theta}_i(t) = \dot{\theta}_j(t)\,,\, \forall i, j\,.$

Kuramoto model

$$\dot{\theta}_i = \omega_i - \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_i - \theta_j)\,, \text{for } i = 1, ..., N\,. \qquad (1)$$

ω_i: natural frequencies.

Y. Kuramoto, Lecture Notes in Physics 39, International Symposium on Mathematical Problems in Theoretical Physics (1975).

Kuramoto model

$$\dot{\theta}_i = \omega_i - \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_i - \theta_j)\,,\text{for } i = 1,...,N\,. \tag{1}$$

ω_i: natural frequencies.

Order parameter

$$re^{i\,\psi} = N^{-1}\sum_{j=1}^{N} e^{i\,\theta_j} \tag{2}$$

Y. Kuramoto, Lecture Notes in Physics 39, International Symposium on Mathematical Problems in Theoretical Physics (1975).

Modelling of Synchronization

Kuramoto model

$$\dot{\theta}_i = \omega_i - \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_i - \theta_j)\,, \text{for } i = 1, ..., N\,. \tag{1}$$

ω_i: natural frequencies.

Order parameter

$$re^{i\,\psi} = N^{-1}\sum_{j=1}^{N} e^{i\,\theta_j} \tag{2}$$

ψ: average phase.

Illustration

$$\dot{\theta}_i = \omega_i - Kr\sin(\psi - \theta_i)\,, \text{for } i = 1, ..., N\,. \tag{3}$$

Y. Kuramoto, Lecture Notes in Physics 39, International Symposium on Mathematical Problems in Theoretical Physics (1975).

Kuramoto model

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{N} b_{ij} \sin(\theta_i - \theta_j)\,, \text{for } i = 1, ..., N\,. \tag{4}$$

ω_i: natural frequencies.
b_{ij}: adjacency matrix.

Good survey Dörfler and Bullo, Automatica **50** (6), 1539-1564, (2014).

Kuramoto model

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{N} b_{ij} \sin(\theta_i - \theta_j)\,, \text{for } i = 1, ..., N\,. \tag{4}$$

ω_i: natural frequencies.
b_{ij}: adjacency matrix.

Multistability

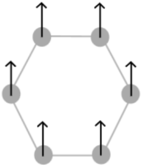
Good survey Dörfler and Bullo, Automatica **50** (6), 1539-1564, (2014).

Kuramoto model

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{N} b_{ij} \sin(\theta_i - \theta_j)\,, \text{for } i = 1, ..., N\,. \tag{4}$$

ω_i: natural frequencies.
b_{ij}: adjacency matrix.

Multistability



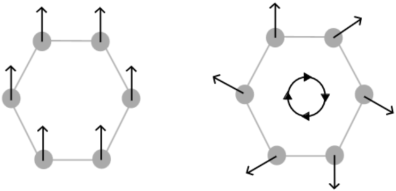
Good survey Dörfler and Bullo, Automatica **50** (6), 1539-1564, (2014).

Kuramoto model

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{N} b_{ij} \sin(\theta_i - \theta_j)\,, \text{for } i = 1, ..., N\,. \tag{4}$$

ω_i: natural frequencies.
b_{ij}: adjacency matrix.

Multistability



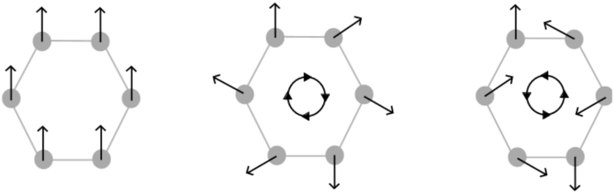
Good survey Dörfler and Bullo, Automatica **50** (6), 1539-1564, (2014)

Kuramoto model

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{N} b_{ij} \sin(\theta_i - \theta_j)\,, \text{for } i = 1, ..., N\,. \tag{4}$$

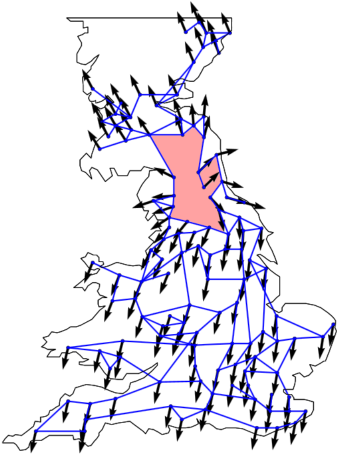
ω_i: natural frequencies.
b_{ij}: adjacency matrix.

Multistability



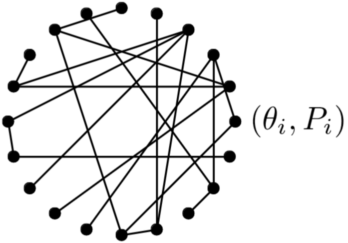
Good survey Dörfler and Bullo, Automatica **50** (6), 1539-1564, (2014)

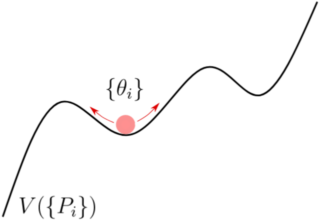
Synchronization on networks



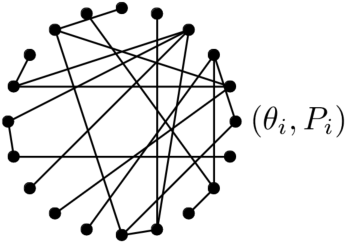
Delabays, MT, Jacquod, Chaos **27**(10), 103109 (2017)

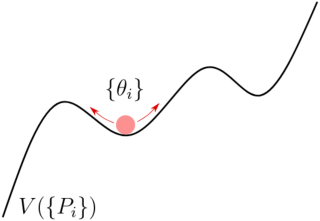
Robustness of synchronous networks





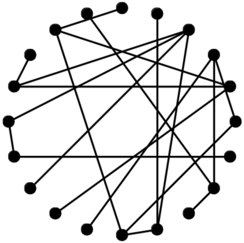
Robustness of synchronous networks

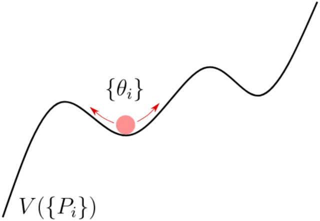




- Size of the basin of attraction

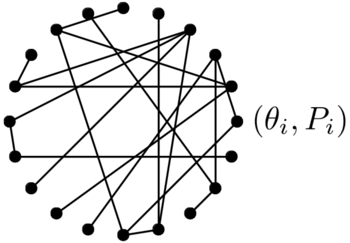
Robustness of synchronous networks

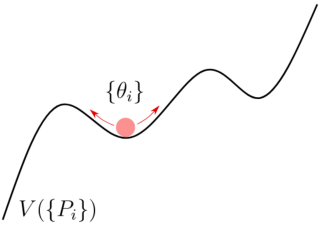




- Size of the basin of attraction
- Near equilibrium dynamics

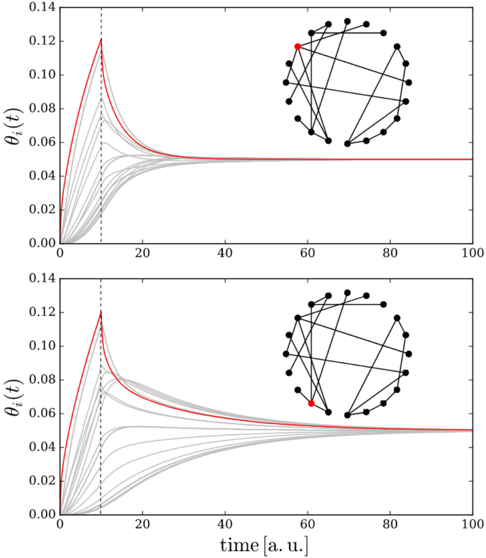
Robustness of synchronous networks





- Size of the basin of attraction
- Near equilibrium dynamics
- Transitions between fixed points

Robustness of synchronous networks



Near equilibrium dynamics

$$0 = \omega_i - \sum_{j=1}^{N} b_{ij} \sin(\theta_i^{(0)} - \theta_j^{(0)}) \,, \text{for } i = 1, ..., N \,. \tag{5}$$

Robustness of synchronous networks to noise inputs

Near equilibrium dynamics

$$0 = \omega_i - \sum_{j=1}^{N} b_{ij} \sin(\theta_i^{(0)} - \theta_j^{(0)})\,, \text{for } i = 1, ..., N\,. \qquad (5)$$

$$\delta\dot{\theta}_i = -\sum_{j=1}^{N} b_{ij} \cos(\theta_i^{(0)} - \theta_j^{(0)})(\delta\theta_i - \delta\theta_j) + \eta_i(t)\,, \text{for } i = 1, ..., N\,. \qquad (6)$$

$\eta_i(t)$: Noise inputs.

Robustness of synchronous networks to noise inputs

Near equilibrium dynamics

$$\boldsymbol{\delta\dot{\theta}} = -\mathbb{L}(\{\theta_k^{(0)}\})\boldsymbol{\delta\theta} + \boldsymbol{\eta}(t)\,, \text{for } i = 1, ..., N\,. \tag{7}$$

$\mathbb{L}(\{\theta_k^{(0)}\})$: Jacobian, with eigenvalues $\lambda_1 = 0 < \lambda_2 \leq ... \leq \lambda_N$.

Robustness of synchronous networks to noise inputs

Near equilibrium dynamics

$$\boldsymbol{\delta\dot{\theta}} = -\mathbb{L}(\{\theta_k^{(0)}\})\boldsymbol{\delta\theta} + \boldsymbol{\eta}(t)\,, \text{for } i = 1, ..., N\,. \tag{7}$$

$\mathbb{L}(\{\theta_k^{(0)}\})$: Jacobian, with eigenvalues $\lambda_1 = 0 < \lambda_2 \leq ... \leq \lambda_N$.

Solution

$$\delta\theta_i(t) = \sum_\alpha e^{-\lambda_\alpha\, t} \int_0^t e^{\lambda_\alpha\, t'} \sum_j \eta_j(t') u_{\alpha,j} \mathrm{d}t'\, u_{\alpha,i}\,. \tag{8}$$

Uncorrelated white noise

$$\langle \eta_i(t)\eta_j(t')\rangle = \eta_0^2\,\tau_0\,\delta_{ij}\,\delta(t-t') \tag{9}$$

Robustness of synchronous networks to noise inputs

Uncorrelated white noise

$$\langle \eta_i(t)\eta_j(t')\rangle = \eta_0^2\,\tau_0\,\delta_{ij}\,\delta(t-t') \tag{9}$$

Variance at node *i*

$$\langle \delta\theta_i^2\rangle = \frac{\eta_0^2\,\tau_0}{2}\sum_\alpha \frac{u_{\alpha,i}^2}{\lambda_\alpha} \tag{10}$$

Robustness of synchronous networks to noise inputs

Uncorrelated white noise

$$\langle \eta_i(t)\eta_j(t')\rangle = \eta_0^2\,\tau_0\,\delta_{ij}\,\delta(t-t') \tag{9}$$

Variance at node *i*

$$\langle \delta\theta_i^2\rangle = \frac{\eta_0^2\,\tau_0}{2}\sum_\alpha \frac{u_{\alpha,i}^2}{\lambda_\alpha} \tag{10}$$

Average variance in the network

$$N^{-1}\sum_i \langle \delta\theta_i^2\rangle = \frac{\eta_0^2\,\tau_0}{2N}\sum_\alpha \frac{1}{\lambda_\alpha} \tag{11}$$

Robustness of synchronous networks to noise inputs

Time-correlated noise

$$\langle \eta_i(t)\eta_j(t')\rangle = \eta_0^2\,\delta_{ij}\,e^{-|t-t'|/\tau_0} \tag{12}$$

Time-correlated noise

$$\langle \eta_i(t)\eta_j(t')\rangle = \eta_0^2\,\delta_{ij}\,e^{-|t-t'|/\tau_0} \tag{12}$$

Variance at node i for $\tau_0 \gg \lambda_\alpha^{-1}$

$$\langle \delta\theta_i^2\rangle = \eta_0^2 \sum_\alpha \frac{u_{\alpha,i}^2}{\lambda_\alpha^2} \tag{13}$$

Time-correlated noise

$$\langle \eta_i(t)\eta_j(t')\rangle = \eta_0^2\, \delta_{ij}\, e^{-|t-t'|/\tau_0} \tag{12}$$

Variance at node i for $\tau_0 \gg \lambda_\alpha^{-1}$

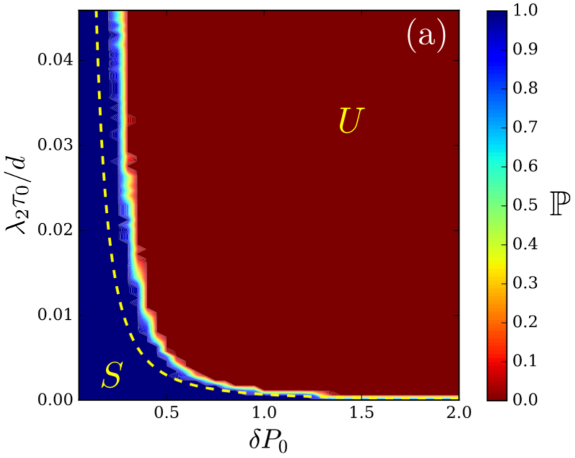
$$\langle \delta\theta_i^2 \rangle = \eta_0^2 \sum_\alpha \frac{u_{\alpha,i}^2}{\lambda_\alpha^2} \tag{13}$$

Average variance in the network

$$N^{-1} \sum_i \langle \delta\theta_i^2 \rangle = \frac{\eta_0^2}{N} \sum_\alpha \frac{1}{\lambda_\alpha^2} \tag{14}$$

Robustness of synchronous networks to noise inputs

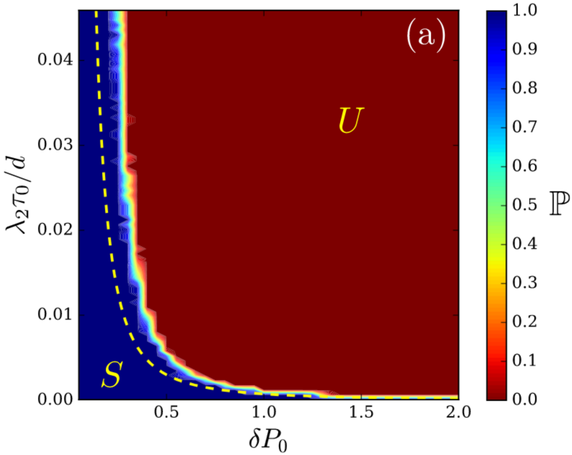
Escape from the initial basin of attraction



MT, Delabays, Jacquod, Phys. Rev. E **99** (6), 062213 (2019)

Robustness of synchronous networks to noise inputs

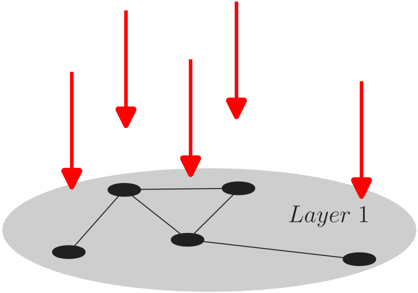
Escape from the initial basin of attraction



It can be even worst!

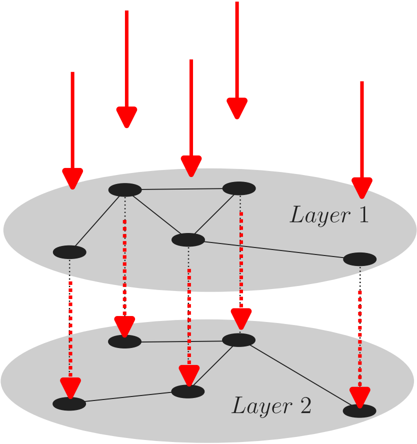
MT, Delabays, Jacquod, Phys. Rev. E **99** (6), 062213 (2019)

Layered Networks



MT, J. Phys. Complex. **3**, 03LT01 (2022)
MT, Chaos **32**(12), 121102, *fast track* (2022)

Layered Networks



MT, J. Phys. Complex. **3**, 03LT01 (2022)
MT, Chaos **32**(12), 121102, *fast track* (2022)

Layered Kuramoto oscillators:

$$\begin{aligned} \dot{\phi}_i &= \omega_i^{(1)} - \sum_{j=1}^{N} b_{ij}^{(1)} \sin(\phi_i - \phi_j) + \eta_i \quad i = 1, ...N\,, \\ \dot{\theta}_i &= \omega_i^{(2)} - \sum_{j=1}^{N} b_{ij}^{(2)} \sin(\theta_i - \theta_j) + f_i(\{\phi_k\}, \{\theta_k\}) \quad i = 1, ...N, , \end{aligned} \tag{15}$$

MT, J. Phys. Complex. **3**, 03LT01 (2022)
MT, Chaos **32**(12), 121102, *fast track* (2022)

Layered Networks: Multistability

Layered Kuramoto oscillators:

$$
\begin{aligned}
\dot{\phi}_i &= \omega_i^{(1)} - \sum_{j=1}^{N} b_{ij}^{(1)} \sin(\phi_i - \phi_j) + \eta_i \quad i = 1, ...N\,, \\
\dot{\theta}_i &= \omega_i^{(2)} - \sum_{j=1}^{N} b_{ij}^{(2)} \sin(\theta_i - \theta_j) + f_i(\{\phi_k\}, \{\theta_k\}) \quad i = 1, ...N, ,
\end{aligned}
\tag{15}
$$

$$
\mathbb{L}_{ij}^{(1)}(\{\phi_i^{(0)}\}) = \begin{cases} -b_{ij}^{(1)} \cos(\phi_i^{(0)} - \phi_j^{(0)})\,, & i \neq j\,, \\ \sum_k b_{ik}^{(1)} \cos(\phi_i^{(0)} - \phi_k^{(0)})\,, & i = j\,, \end{cases}
$$

$$
\mathbb{L}_{ij}^{(2)}(\{\theta_i^{(0)}\}) = \begin{cases} -b_{ij}^{(2)} \cos(\theta_i^{(0)} - \theta_j^{(0)})\,, & i \neq j\,, \\ \sum_k b_{ik}^{(2)} \cos(\theta_i^{(0)} - \theta_k^{(0)})\,, & i = j\,, \end{cases}
$$

Layered Kuramoto oscillators:

$$\begin{aligned}\dot{\phi}_i &= \omega_i^{(1)} - \sum_{j=1}^{N} b_{ij}^{(1)} \sin(\phi_i - \phi_j) + \eta_i \quad i = 1, ...N\,, \\ \dot{\theta}_i &= \omega_i^{(2)} - \sum_{j=1}^{N} b_{ij}^{(2)} \sin(\theta_i - \theta_j) + f_i(\{\phi_k\}, \{\theta_k\}) \quad i = 1, ...N, ,\end{aligned} \tag{15}$$

Two sets of time-scales:

$$\lambda_1^{(1)} = 0 < \lambda_2^{(1)} \leq ... \leq \lambda_N^{(1)} \tag{16}$$

$$\lambda_1^{(2)} = 0 < \lambda_2^{(2)} \leq ... \leq \lambda_N^{(2)} \tag{17}$$

Layered Networks: Multistability

Layered Kuramoto oscillators:

$$\dot{\phi}_i = \omega_i^{(1)} - \sum_{j=1}^{N} b_{ij}^{(1)} \sin(\phi_i - \phi_j) + \eta_i \quad i = 1, ...N \,,$$
$$\dot{\theta}_i = \omega_i^{(2)} - \sum_{j=1}^{N} b_{ij}^{(2)} \sin(\theta_i - \theta_j) + f_i(\{\phi_k\}, \{\theta_k\}) \quad i = 1, ...N, , \tag{15}$$

Uncorrelated white noise: $\langle \eta_i(t) \eta_j(t') \rangle = \eta_0^2 \, \delta_{ij} \, \delta(t - t')$

MT, J. Phys. Complex. **3**, 03LT01 (2022)
MT, Chaos **32**(12), 121102, *fast track* (2022)

Layered Kuramoto oscillators:

$$\begin{aligned}\dot{\phi}_i &= \omega_i^{(1)} - \sum_{j=1}^{N} b_{ij}^{(1)} \sin(\phi_i - \phi_j) + \eta_i \quad i = 1, \ldots N\,, \\ \dot{\theta}_i &= \omega_i^{(2)} - \sum_{j=1}^{N} b_{ij}^{(2)} \sin(\theta_i - \theta_j) + f_i(\{\phi_k\}, \{\theta_k\}) \quad i = 1, \ldots N, ,\end{aligned} \tag{15}$$

Uncorrelated white noise: $\langle \eta_i(t)\eta_j(t')\rangle = \eta_0^2\, \delta_{ij}\, \delta(t - t')$

Simplest choice: $f_i(\{\phi_k\}, \{\theta_k\}) = d(\phi_i - N^{-1} \sum_j \phi_j)$

MT, J. Phys. Complex. **3**, 03LT01 (2022)
MT, Chaos **32**(12), 121102, *fast track* (2022)

Layered Kuramoto oscillators:

$$\begin{aligned} \dot{\phi}_i &= \omega_i^{(1)} - \sum_{j=1}^{N} b_{ij}^{(1)} \sin(\phi_i - \phi_j) + \eta_i \quad i = 1, ...N\,, \\ \dot{\theta}_i &= \omega_i^{(2)} - \sum_{j=1}^{N} b_{ij}^{(2)} \sin(\theta_i - \theta_j) + f_i(\{\phi_k\}, \{\theta_k\}) \quad i = 1, ...N, , \end{aligned} \tag{15}$$

Uncorrelated white noise: $\langle \eta_i(t)\eta_j(t')\rangle = \eta_0^2\, \delta_{ij}\, \delta(t - t')$

Simplest choice: $f_i(\{\phi_k\}, \{\theta_k\}) = d(\phi_i - N^{-1} \sum_j \phi_j)$

Noise in the 2nd layer: $\langle \phi_i(t)\phi_j(t')\rangle = \frac{\eta_0^2}{2} \sum_\alpha \frac{u_{\alpha,i}^{(1)} u_{\alpha,j}^{(1)}}{\lambda_\alpha^{(1)}} e^{-\lambda_\alpha^{(1)}|t-t'|}$.

MT, J. Phys. Complex. **3**, 03LT01 (2022)
MT, Chaos **32**(12), 121102, *fast track* (2022)

Layered Networks

Analytical treatment:

$$\phi_i(t) = \sum_\alpha e^{-\lambda_\alpha^{(1)} t} \int_0^t e^{\lambda_\alpha^{(1)} t'} \sum_j \eta_j \, u_{\alpha,j}^{(1)} \, \mathrm{d}t' \, u_{\alpha,i}^{(1)} \,, \tag{16}$$

$$\theta_i(t) = \sum_\alpha e^{-\lambda_\alpha^{(2)} t} \int_0^t e^{\lambda_\alpha^{(2)} t'} \sum_j \phi_j \, u_{\alpha,j}^{(2)} \, \mathrm{d}t' \, u_{\alpha,i}^{(2)} \,. \tag{17}$$

Layered Networks

Analytical treatment:

$$\phi_i(t) = \sum_\alpha e^{-\lambda_\alpha^{(1)} t} \int_0^t e^{\lambda_\alpha^{(1)} t'} \sum_j \eta_j \, u_{\alpha,j}^{(1)} \, \mathrm{d}t' \, u_{\alpha,i}^{(1)} \,, \tag{16}$$

$$\theta_i(t) = \sum_\alpha e^{-\lambda_\alpha^{(2)} t} \int_0^t e^{\lambda_\alpha^{(2)} t'} \sum_j \phi_j \, u_{\alpha,j}^{(2)} \, \mathrm{d}t' \, u_{\alpha,i}^{(2)} \,. \tag{17}$$

Layer 1:

$$\langle \phi_i^2 \rangle = \frac{\eta_0^2}{2} \sum_\alpha \frac{{u_{\alpha,i}^{(1)}}^2}{\lambda_\alpha^{(1)}} \,, \tag{18}$$

Analytical treatment:

$$\phi_i(t) = \sum_\alpha e^{-\lambda_\alpha^{(1)} t} \int_0^t e^{\lambda_\alpha^{(1)} t'} \sum_j \eta_j \, u_{\alpha,j}^{(1)} \, \mathrm{d}t' \, u_{\alpha,i}^{(1)} \,, \tag{16}$$

$$\theta_i(t) = \sum_\alpha e^{-\lambda_\alpha^{(2)} t} \int_0^t e^{\lambda_\alpha^{(2)} t'} \sum_j \phi_j \, u_{\alpha,j}^{(2)} \, \mathrm{d}t' \, u_{\alpha,i}^{(2)} \,. \tag{17}$$

Layer 1:

$$\langle \phi_i^2 \rangle = \frac{\eta_0^2}{2} \sum_\alpha \frac{{u_{\alpha,i}^{(1)}}^2}{\lambda_\alpha^{(1)}} \,, \tag{18}$$

Layer 2:

$$\langle \theta_i^2 \rangle = \frac{\eta_0^2}{2} \sum_{\alpha,\beta,\gamma} \sum_{k,l} \frac{u_{\gamma,k}^{(1)} u_{\gamma,l}^{(1)} u_{\alpha,k}^{(2)} u_{\beta,l}^{(2)} [2\lambda_\gamma^{(1)} + \lambda_\alpha^{(2)} + \lambda_\beta^{(2)}]}{\lambda_\gamma^{(1)} (\lambda_\alpha^{(2)} + \lambda_\beta^{(2)})(\lambda_\gamma^{(1)} + \lambda_\alpha^{(2)})(\lambda_\gamma^{(1)} + \lambda_\beta^{(2)})} u_{\alpha,i}^{(2)} u_{\beta,i}^{(2)} \,. \tag{19}$$

Layered Networks

Analytical treatment:

$$\phi_i(t) = \sum_\alpha e^{-\lambda_\alpha^{(1)} t} \int_0^t e^{\lambda_\alpha^{(1)} t'} \sum_j \eta_j \, u_{\alpha,j}^{(1)} \, \mathrm{d}t' \, u_{\alpha,i}^{(1)} \, , \tag{16}$$

$$\theta_i(t) = \sum_\alpha e^{-\lambda_\alpha^{(2)} t} \int_0^t e^{\lambda_\alpha^{(2)} t'} \sum_j \phi_j \, u_{\alpha,j}^{(2)} \, \mathrm{d}t' \, u_{\alpha,i}^{(2)} \, . \tag{17}$$

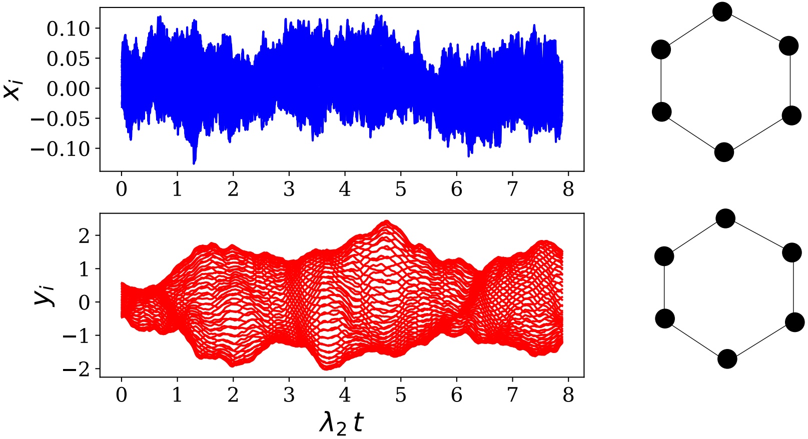
Layer 1:

$$\langle \phi_i^2 \rangle = \frac{\eta_0^2}{2} \sum_\alpha \frac{{u_{\alpha,i}^{(1)}}^2}{\lambda_\alpha^{(1)}} \, , \tag{18}$$

Layer 2: Same networks

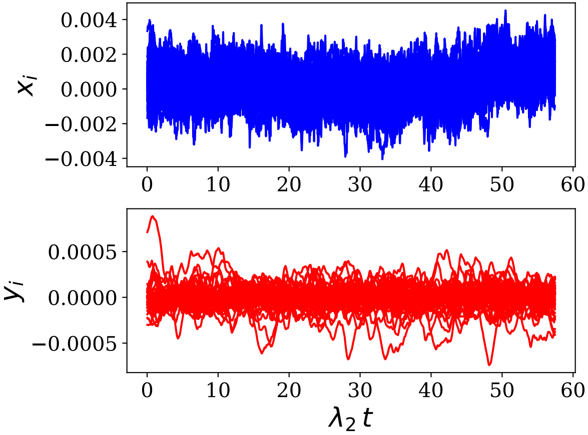
$$\langle \theta_i^2 \rangle = \frac{\eta_0^2}{4} \sum_\alpha \frac{u_{\alpha,i}^2}{\lambda_\alpha^3} \, . \tag{19}$$

Layered Networks: Amplification



MT, J. Phys. Complex. **3**, 03LT01 (2022)

Layered Networks: Amplification



MT, J. Phys. Complex. **3**, 03LT01 (2022)

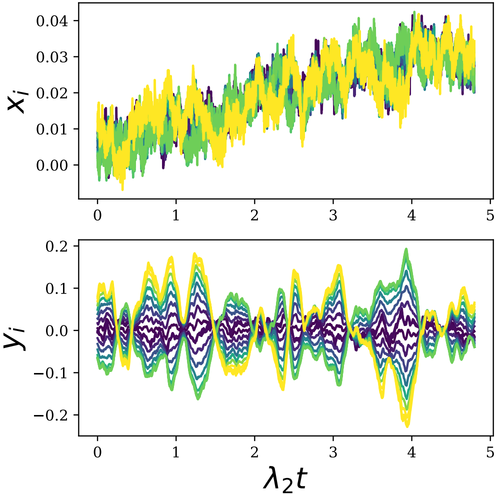
Layered Networks: Transitions

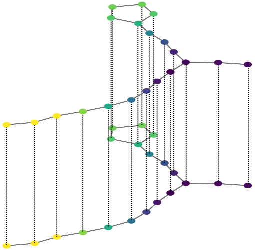
$$\langle x_i^2 \rangle = \frac{\eta_0^2}{2} \sum_\alpha \frac{{u_{\alpha,i}^{(1)}}^2}{\lambda_\alpha^{(1)}} \, , \langle y_i^2 \rangle = \frac{\eta_0^2}{4} \sum_\alpha \frac{u_{\alpha,i}^2}{\lambda_\alpha^3} \, .$$

MT, Chaos **32**(12), 121102, *fast track* (2022)

Layered Networks: Transitions

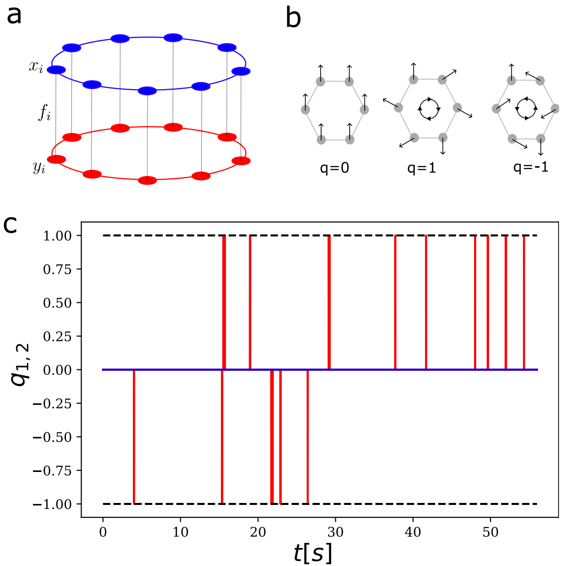
$$\langle x_i^2 \rangle = \frac{\eta_0^2}{2} \sum_\alpha \frac{{u_{\alpha,i}^{(1)}}^2}{\lambda_\alpha^{(1)}} \, , \langle y_i^2 \rangle = \frac{\eta_0^2}{4} \sum_\alpha \frac{u_{\alpha,i}^2}{\lambda_\alpha^3} \, .$$





MT, Chaos **32**(12), 121102, *fast track* (2022)

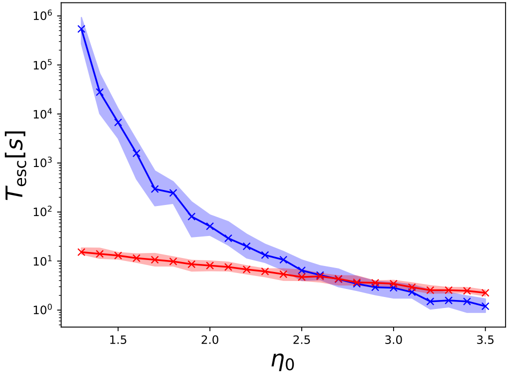
Layered Networks: Transitions



MT, Chaos **32**(12), 121102, *fast track* (2022)

Layered Networks: Transitions

Cycle



MT, Chaos **32**(12), 121102, *fast track* (2022)

Layered Networks: Transitions

Rescaled noise: $\boldsymbol{\eta} = d\,\overline{\phi}$

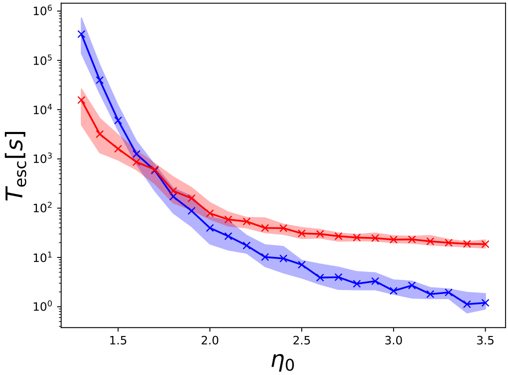
$$N^{-1}\sum_i \langle \xi_i^2 \rangle = \eta_0^2\,, \tag{20}$$

MT, Chaos **32**(12), 121102, *fast track* (2022)

Layered Networks: Transitions

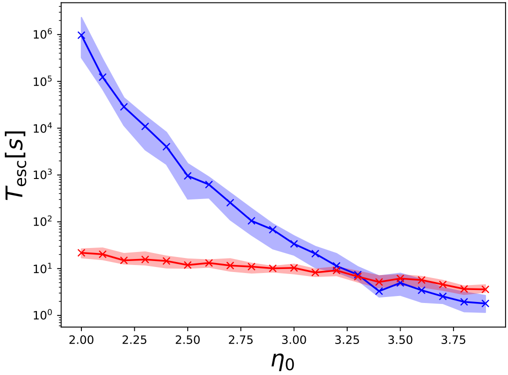
Rescaled noise: $\boldsymbol{\eta} = d\,\overline{\boldsymbol{\phi}}$

$$N^{-1}\sum_i \langle \xi_i^2 \rangle = \eta_0^2\,, \tag{20}$$



MT, Chaos **32**(12), 121102, *fast track* (2022)

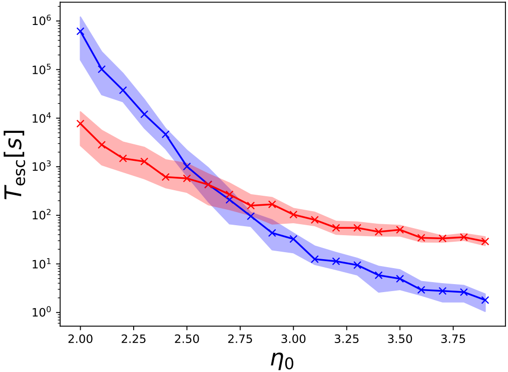
Watts-Strogatz



MT, Chaos **32**(12), 121102, *fast track* (2022)

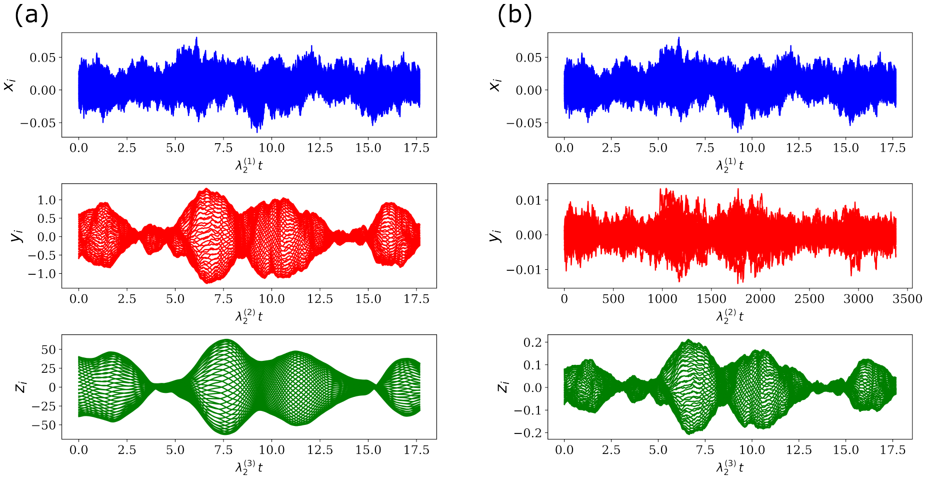
Rescaled noise: $\eta = d\,\overline{\delta \mathbf{x}}$

$$N^{-1}\sum_i \langle \xi_i^2 \rangle = \eta_0^2\,, \tag{21}$$



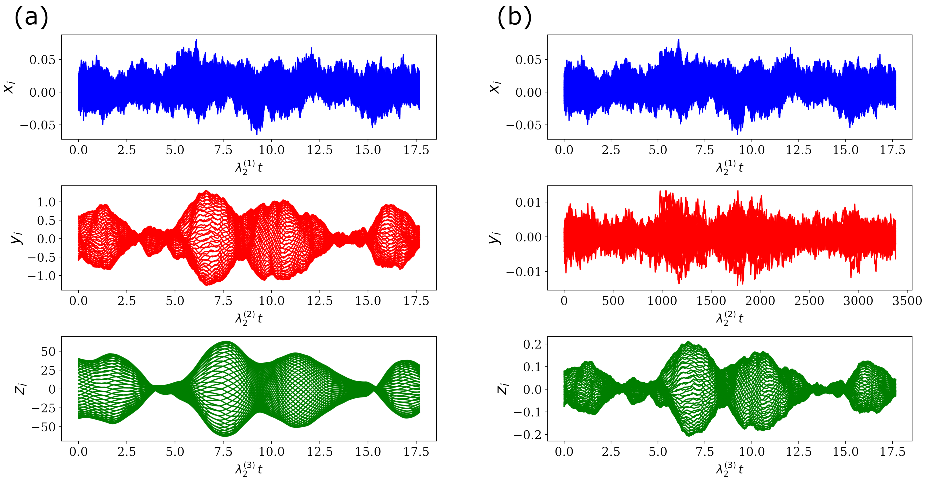
MT, Chaos **32**(12), 121102, *fast track* (2022)

Layered Networks: Amplification



MT, J. Phys. Complex. **3**, 03LT01 (2022)

Layered Networks: Amplification



Applications to photonics and brain dynamics

MT, J. Phys. Complex. **3**, 03LT01 (2022)

Conclusion and future work

So far

- White-noise is not that dangerous... compared to correlated noise.
- Be careful with system-specific correlations.

Conclusion and future work

So far

- White-noise is not that dangerous... compared to correlated noise.
- Be careful with system-specific correlations.

Future work

- Spatial correlation (Collaboration with NRL).