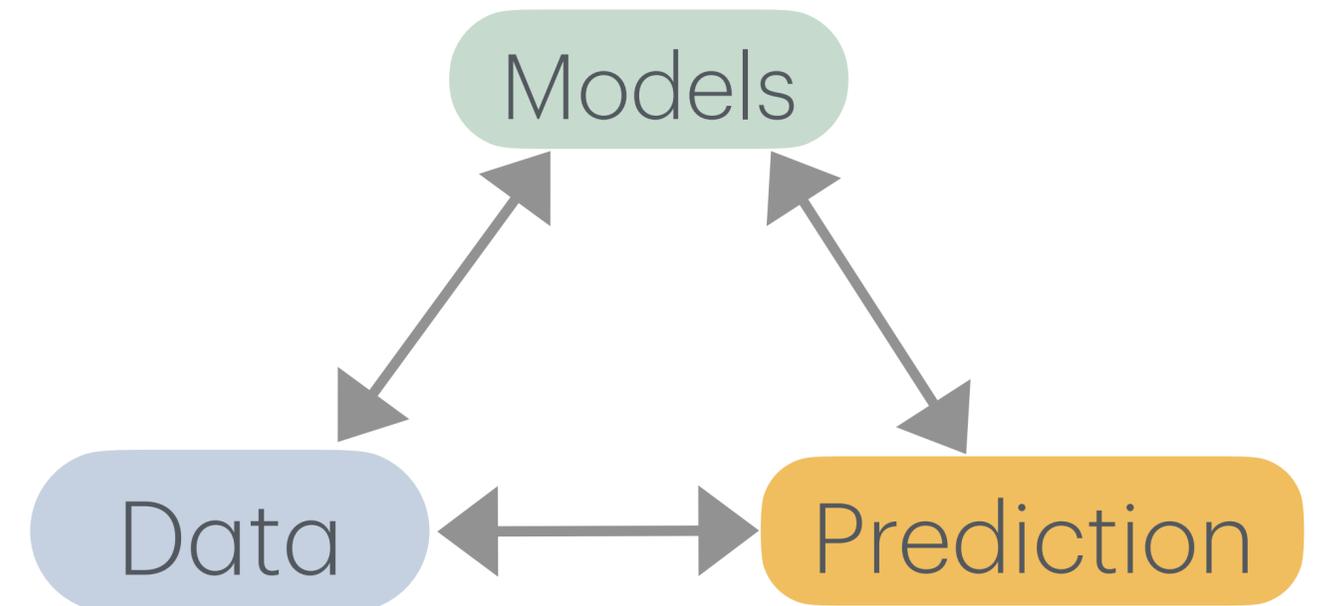


How to pick the right mathematical model? Using optimal control for closed-loop model discrimination.

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The Team



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Laurent Pagnier

Work almost submitted...

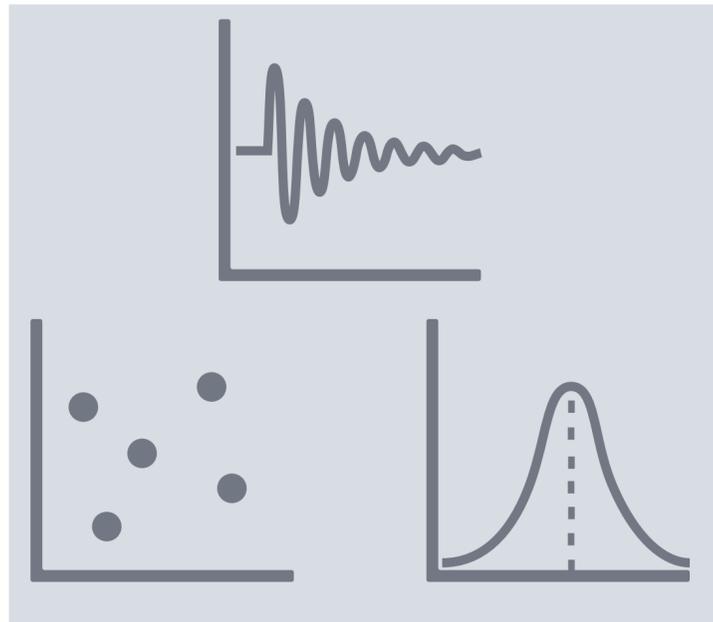
Why model discrimination?

What control?

Control the activity of
networks of neurons

Data — Models

Data



- Measurements from an experiment
- Observation of a process

Models

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X})$$

$$\mathbf{X} \sim \mathcal{P}(\theta)$$

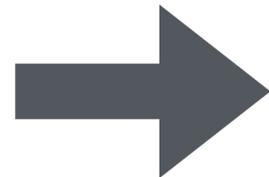
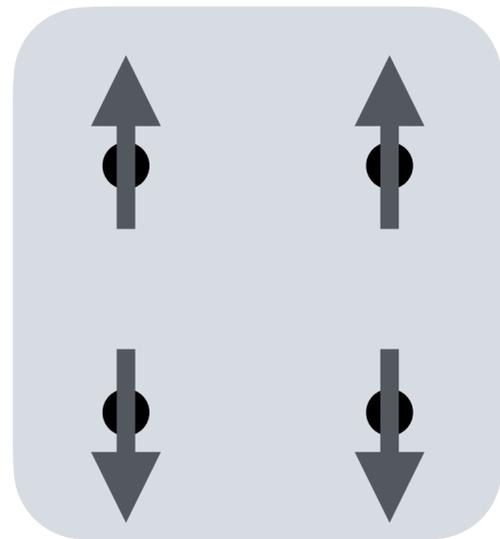
- Differential equations
- Probabilistic models

Explain

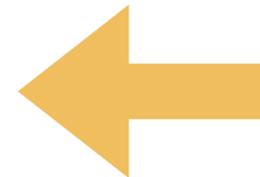
Predict

More than one model...

Observed system



Average magnetization



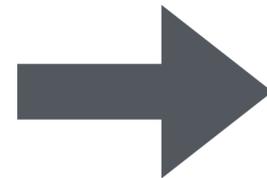
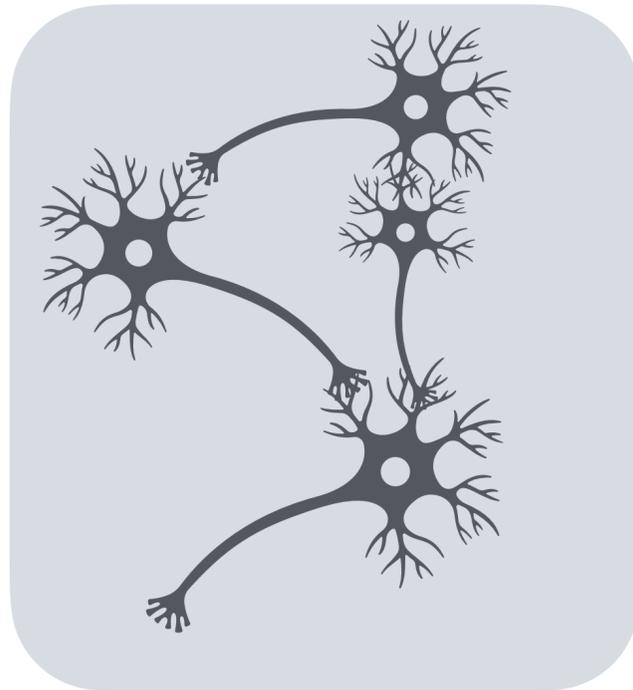
Models

$$\mathcal{H} = \sum_{\langle i,j \rangle} J_{ij} \sigma_i \sigma_j$$

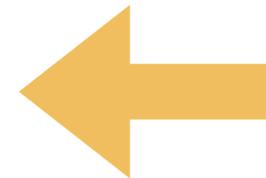
$$\mathcal{H} = \sum_{\langle i,j \rangle} T_{ij} c_{\sigma}^{\dagger} c_{\sigma} + \alpha \sum_i n_{\sigma}^{\dagger} n_{\sigma}$$

More than one model...

Observed system



Spike times



Models

$$\tau \dot{V}_i = V_i^2 + \eta_i + \sum_{j=1}^N A_{ij}(V_i - V_j) + I_i^{\text{ext}}(t),$$

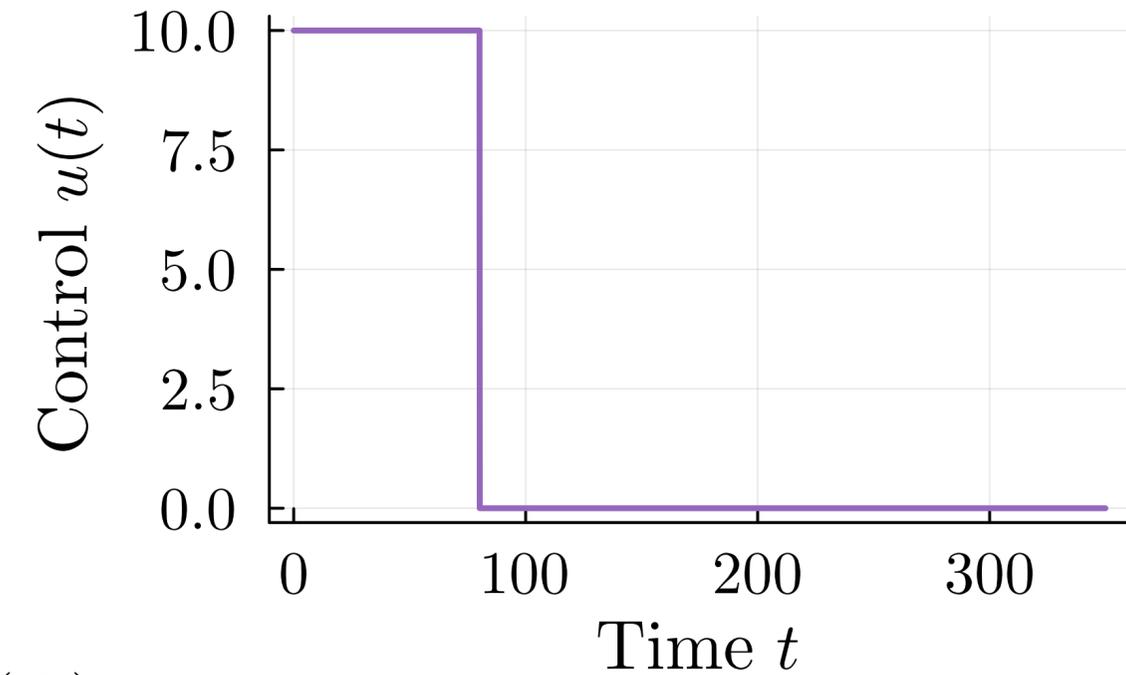
$$V_i(t) = V_{\text{th}} \rightarrow V_i(t) = -V_{\text{th}}$$

$$\dot{V}_i = V_i - \frac{V_i^3}{3} - w + \alpha I_i^{\text{ext}}(t),$$
$$\tau \dot{w}_i = V_i + a - bw_i$$

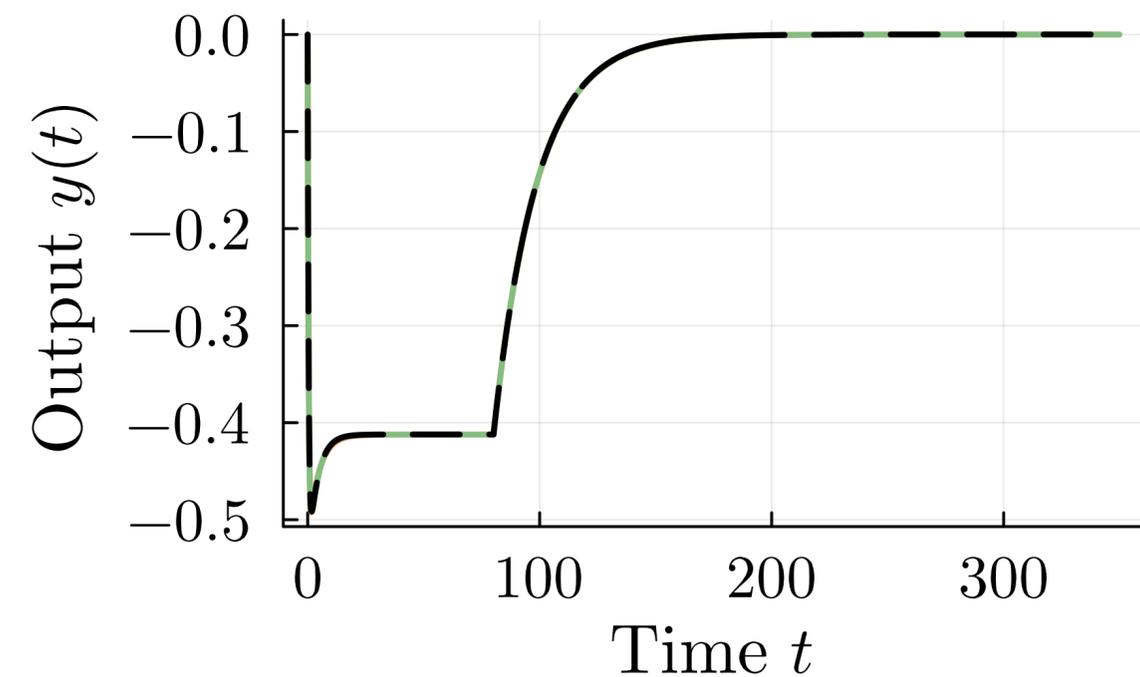
Candidate models

$$\dot{\mathbf{X}}_k = \mathbf{F}_k(\mathbf{X}_k, u(t); \Theta_k),$$
$$\mathbf{Y}_k = \mathbf{O}_k(\mathbf{X}_k; \Theta_k), \quad k = 1, \dots, N.$$

(Ia)



(Ib)



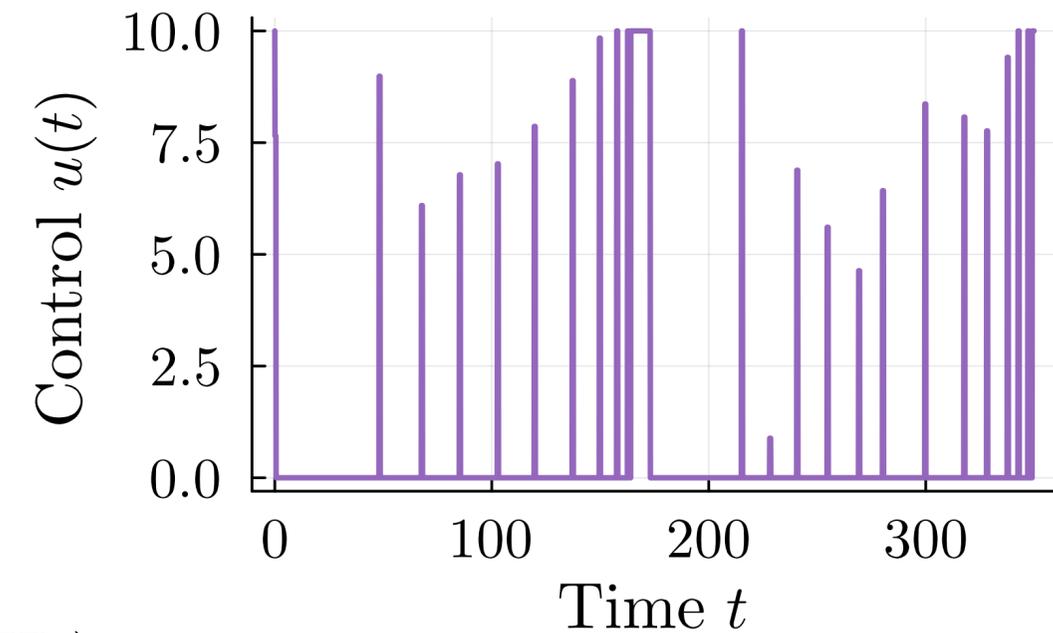
Candidate models

$$\dot{\mathbf{X}}_k = \mathbf{F}_k(\mathbf{X}_k, u(t); \Theta_k),$$
$$\mathbf{Y}_k = \mathbf{O}_k(\mathbf{X}_k; \Theta_k), \quad k = 1, \dots, N.$$

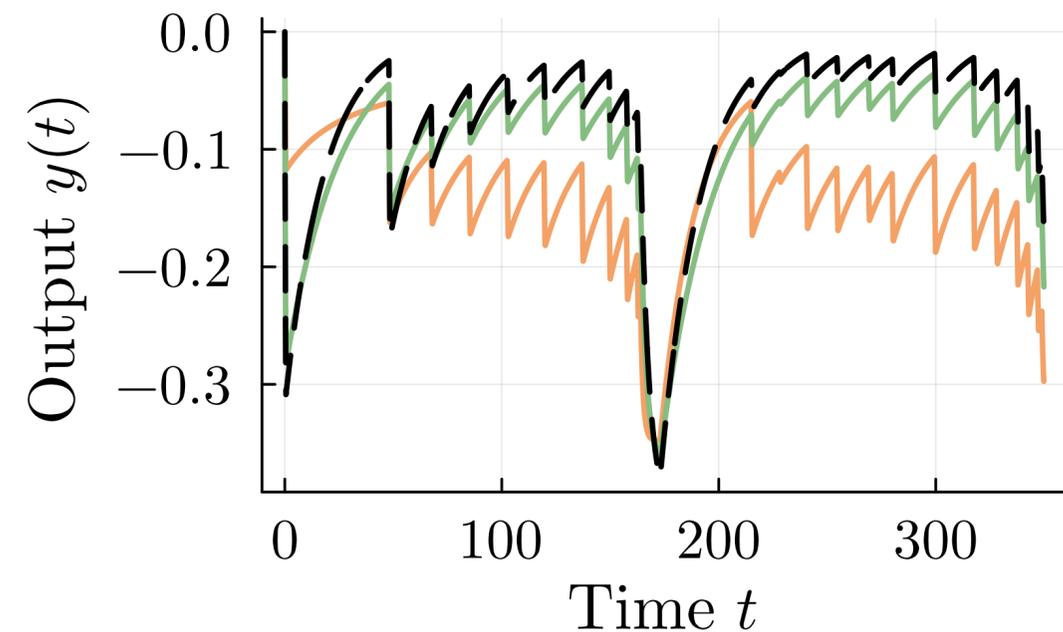


Optimal control

(IIa)

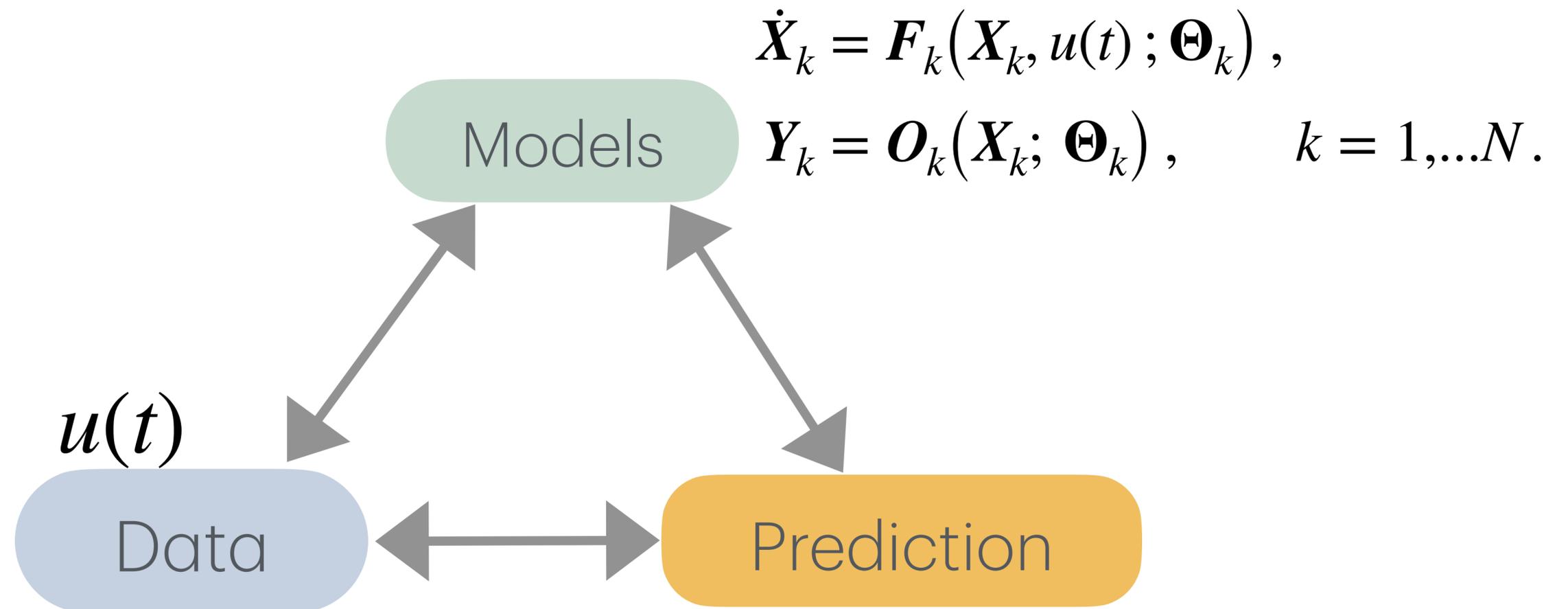


(IIb)



The objective

- For a set of candidate models, find control inputs that allow to choose the right model.



Step 0

Find some candidate models!

$$\dot{\mathbf{X}}_k = \mathbf{F}_k(\mathbf{X}_k, u(t); \Theta_k),$$

$$\mathbf{Y}_k = \mathbf{O}_k(\mathbf{X}_k; \Theta_k), \quad k = 1, 2.$$

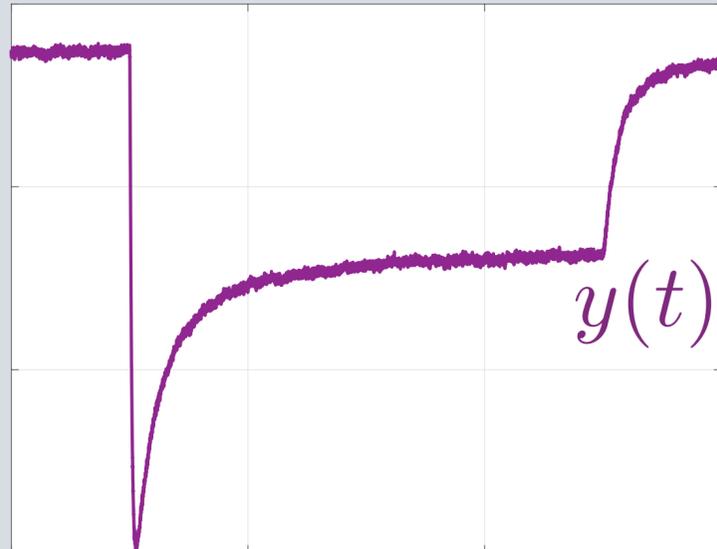
Step 1



Step 2

Parameter fitting

Collect some data



$$\dot{\mathbf{X}}_k = \mathbf{F}_k(\mathbf{X}_k, u(t); \Theta_k),$$

$$\mathbf{Y}_k = \mathbf{O}_k(\mathbf{X}_k; \Theta_k), \quad k = 1, 2.$$

Fitting using a loss function

$$\mathcal{L}(\Theta_k; k) = \int_0^T D(\mathbf{Z}(t), \mathbf{Y}(\mathbf{X}_k(t))) dt$$

$$D(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_2^2$$



Step 3

Optimal Control

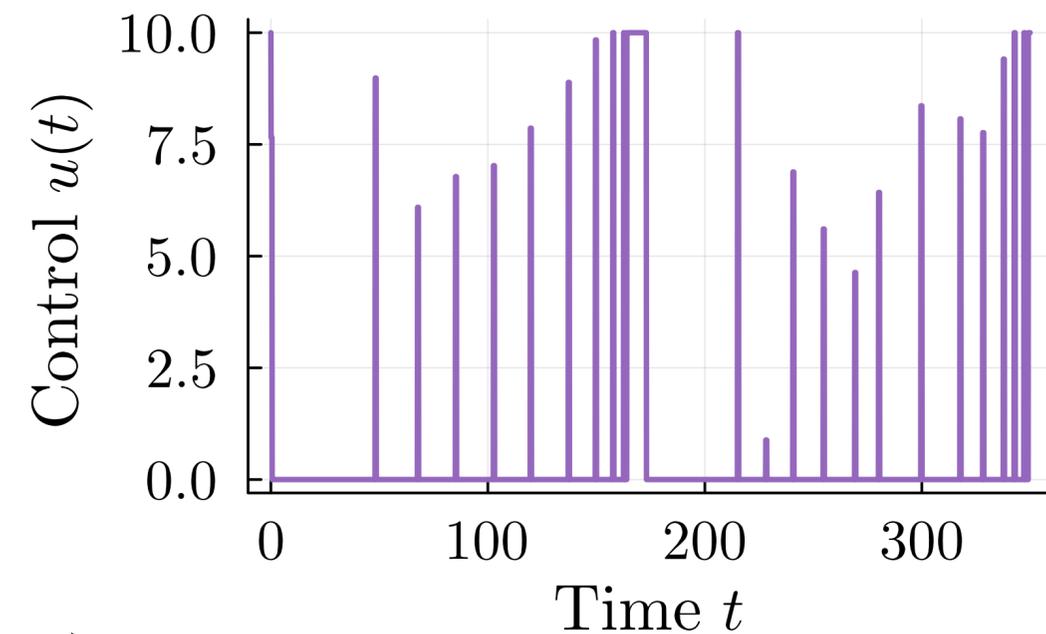
$$\mathcal{J}(u(t)) := \int_0^T \left[D\left(Y_1(X_1(u)), Y_2(X_2(u)) \right) - C(u) \right] dt$$

$$u^* = \operatorname{argmax}_u \mathcal{J}(u)$$

Constraint on past control signals!

Efficiently solved using Ipopt.

(IIa)



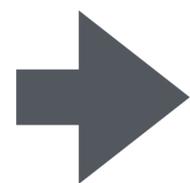
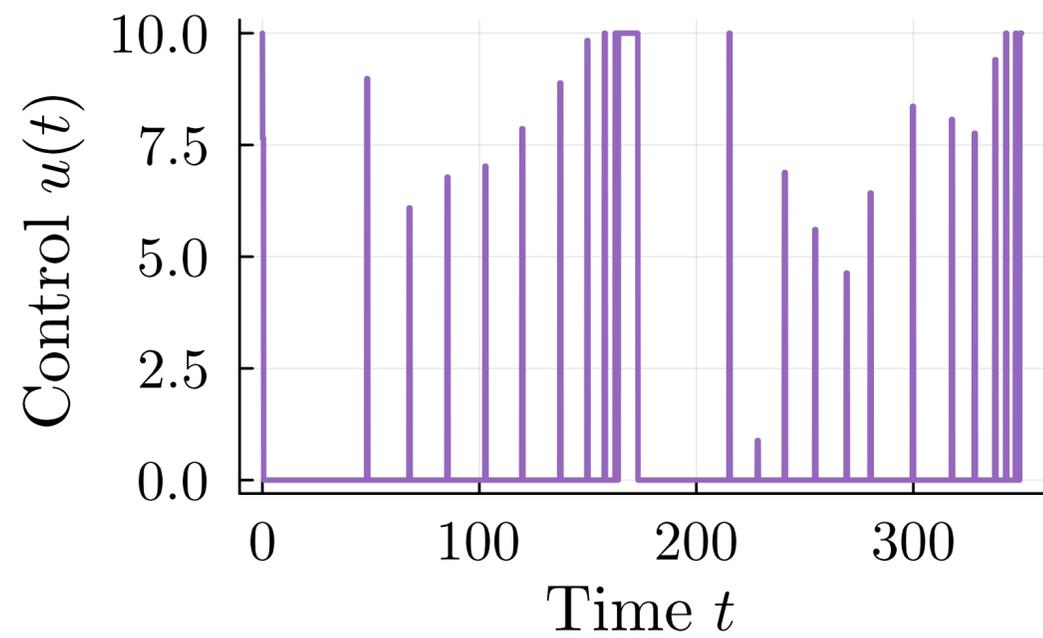


Step 4

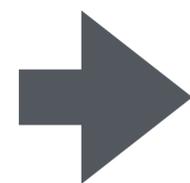


Step 0

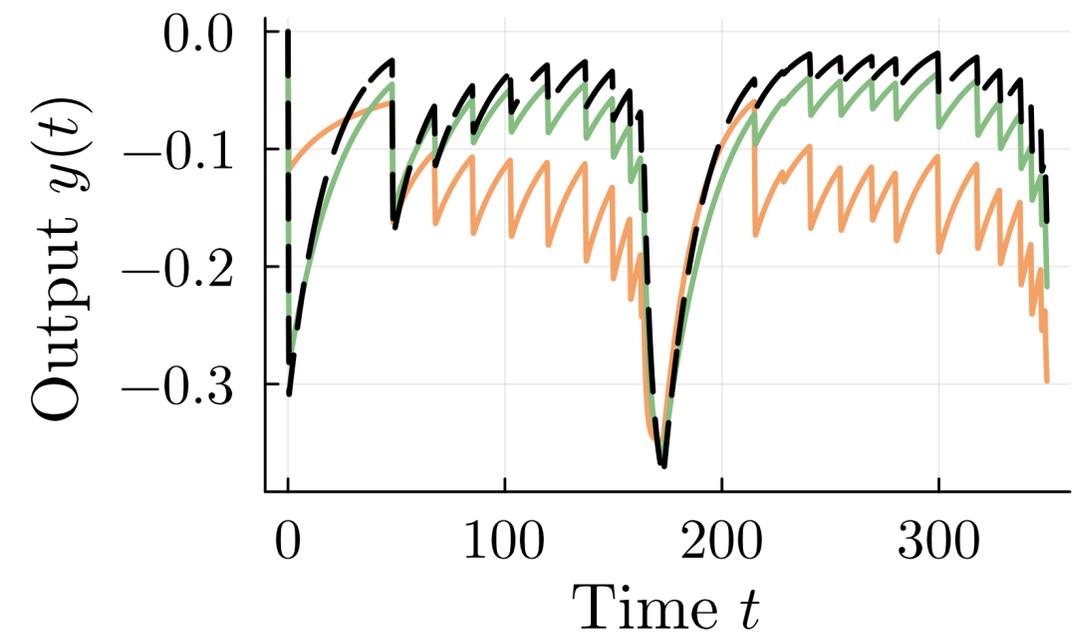
(IIa)



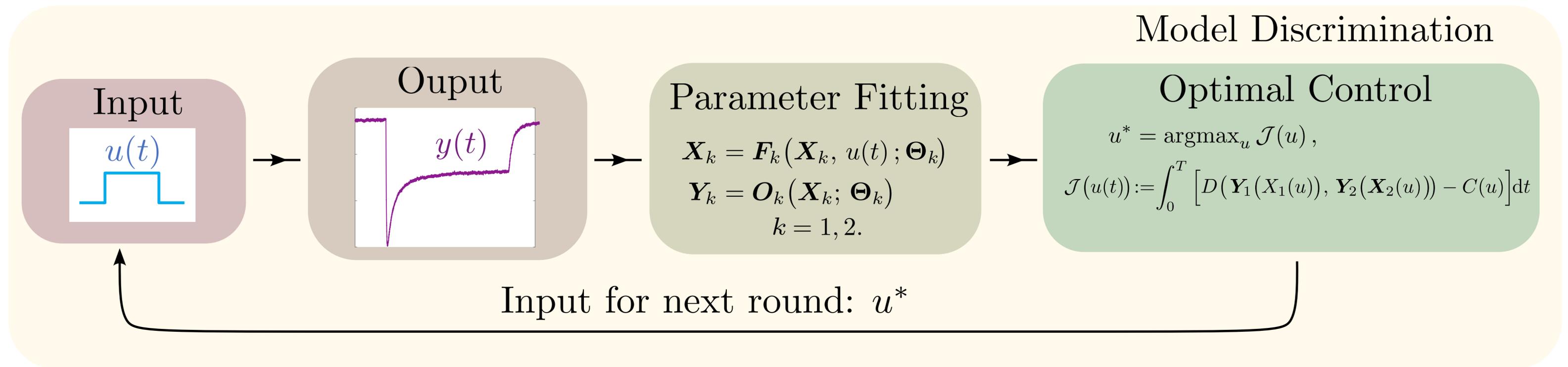
Reference system



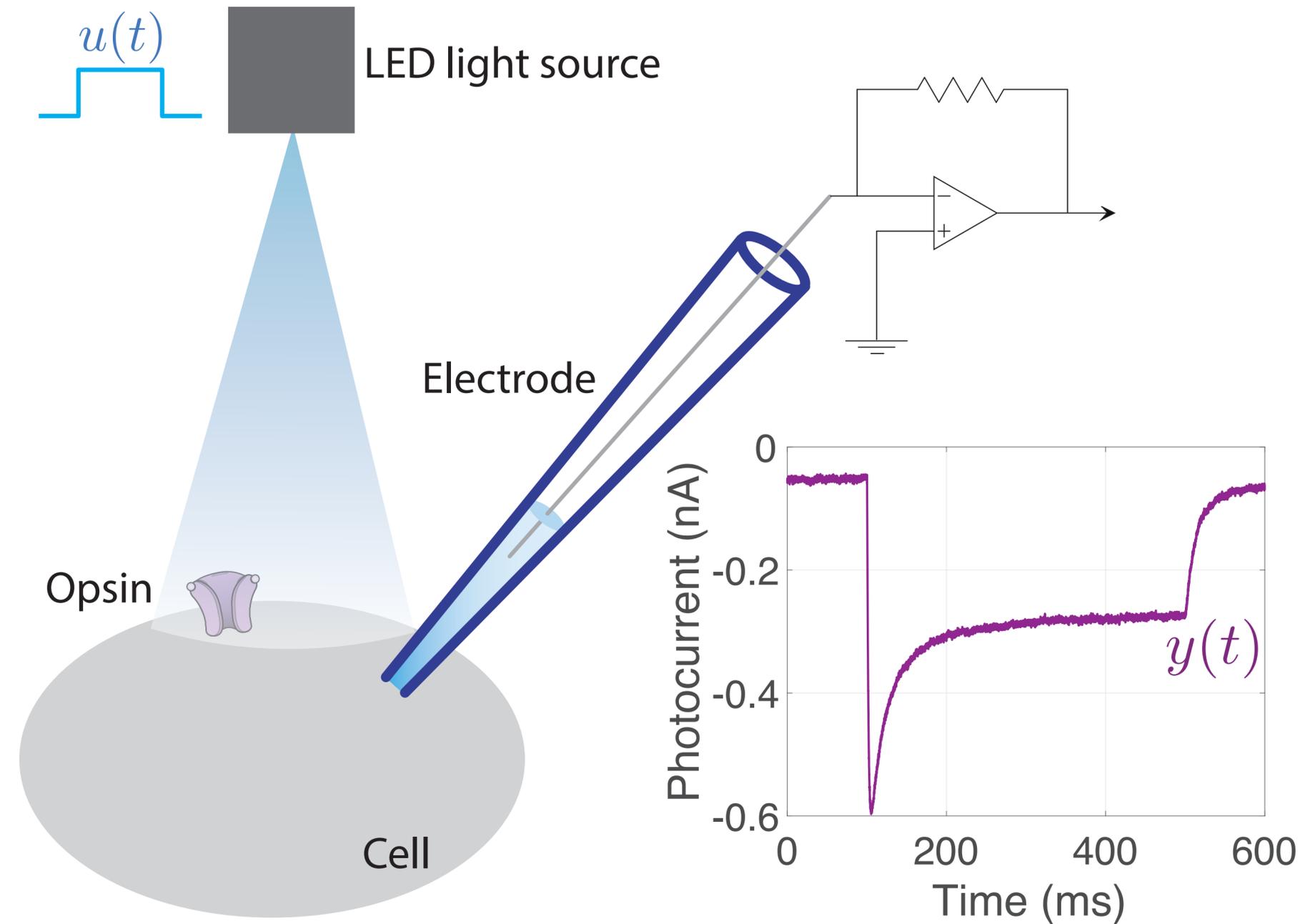
(IIb)



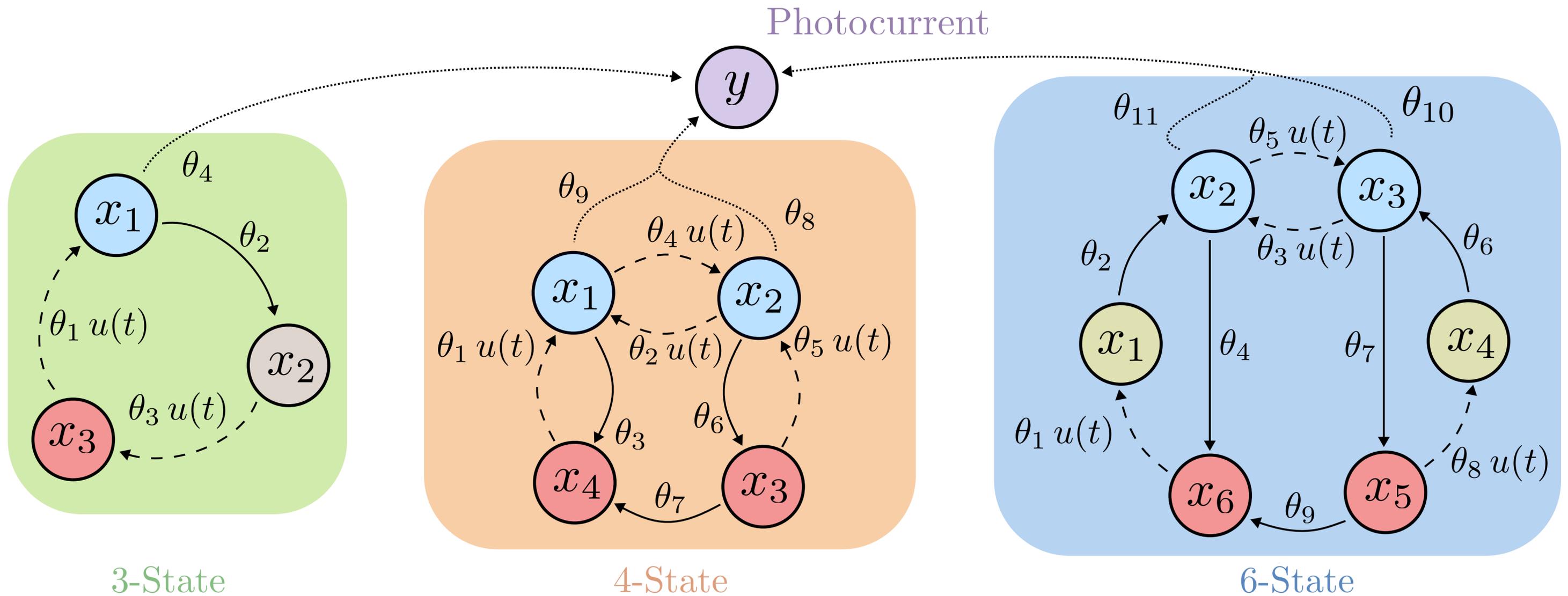
Method Workflow



Electrophysiology experiment



Opsin models



Opsin models

3-state model

$$\begin{aligned}\dot{x}_1 &= \theta_1 u(t) x_3 - \theta_2 x_1, \\ \dot{x}_2 &= \theta_2 x_1 - \theta_3 u(t) x_2, \\ x_3 &= 1 - x_1 - x_2, \\ y &= \theta_4 x_1.\end{aligned}$$

4-state model

$$\begin{aligned}\dot{x}_1 &= \theta_1 u(t) x_4 + \theta_2 u(t) x_2 - [\theta_3 + \theta_4 u(t)] x_1, \\ \dot{x}_2 &= \theta_5 u(t) x_3 + \theta_4 u(t) x_1 - [\theta_6 + \theta_2 u(t)] x_2, \\ \dot{x}_3 &= \theta_6 x_2 - [\theta_7 + \theta_5 u(t)] x_3, \\ x_4 &= 1 - x_1 - x_2 - x_3, \\ y &= \theta_9 x_1 + \theta_8 x_2.\end{aligned}$$

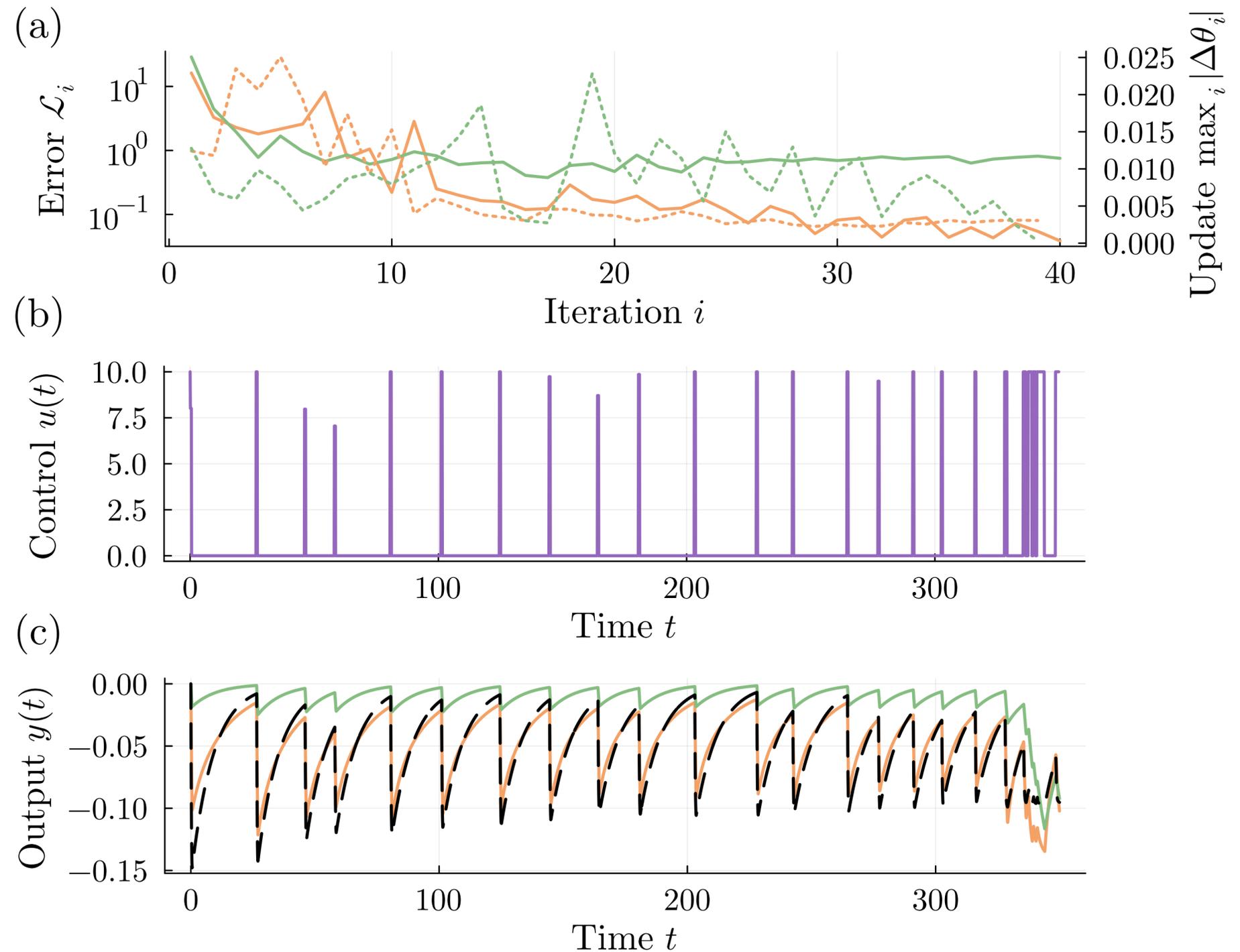
6-state model

$$\begin{aligned}\dot{x}_1 &= \theta_1 u(t) x_6 - \theta_2 x_1, \\ \dot{x}_2 &= \theta_2 x_1 + \theta_3 u(t) x_3 - [\theta_4 + \theta_5 u(t)] x_2, \\ \dot{x}_3 &= \theta_6 x_4 + \theta_5 u(t) x_2 - [\theta_7 + \theta_3 u(t)] x_3, \\ \dot{x}_4 &= \theta_8 u(t) x_5 - \theta_6 x_4, \\ \dot{x}_5 &= \theta_7 x_3 - [\theta_9 + \theta_8 u(t)] x_5, \\ x_6 &= 1 - x_1 - x_2 - x_3 - x_4 - x_5, \\ y &= \theta_{11} x_2 + \theta_{10} x_3.\end{aligned}$$

Numerical Simulations

Candidate models:
3-state vs. 4-state

Reference system:
4-state



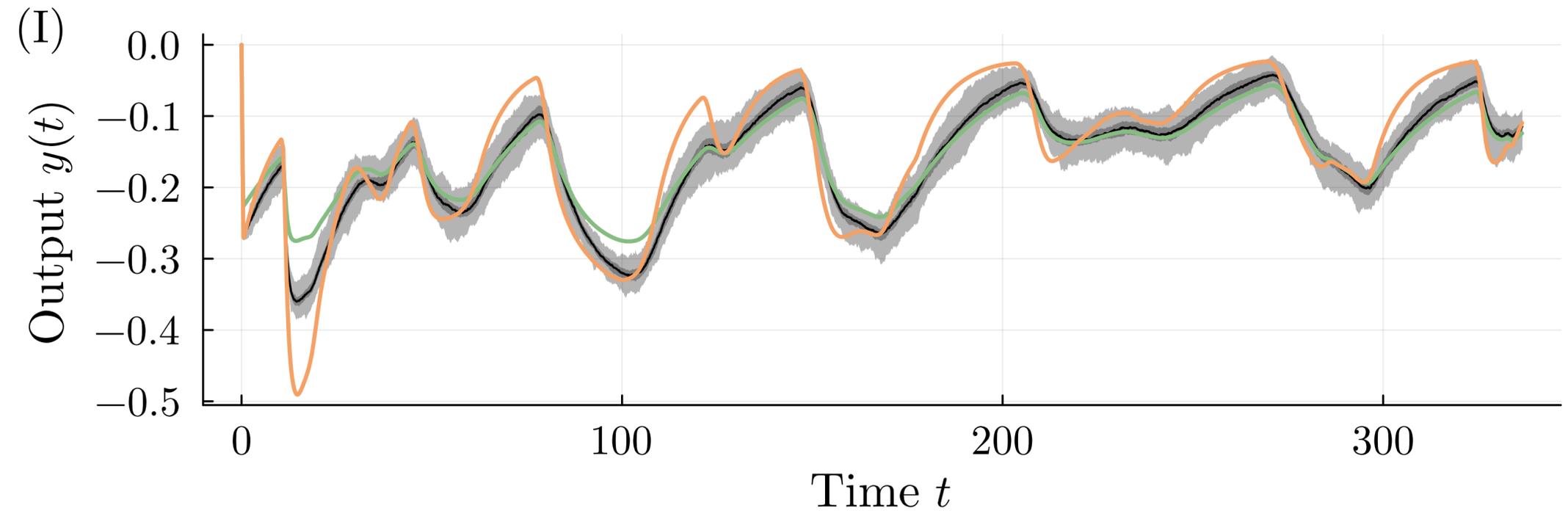
Numerical Simulations

Ref: 3-State	4-State	6-State
3-State	0.090(0.182) 1.334(0.181)	0.432(0.736) 3.393(3.465)
4-State	×	0.978(0.492) 1.281(0.491)
Ref: 4-State	4-State	6-State
3-State	0.320(0.176) 0.228(0.118)	0.469(0.169) 0.284(0.094)
4-State	×	0.230(0.193) 1.044(2.463)
Ref: 6-State	4-State	6-State
3-State	0.571(0.338) 0.226(0.132)	0.473(0.255) 0.279(0.142)
4-State	×	0.725(1.287) 0.718(1.414)

Numerical Simulations

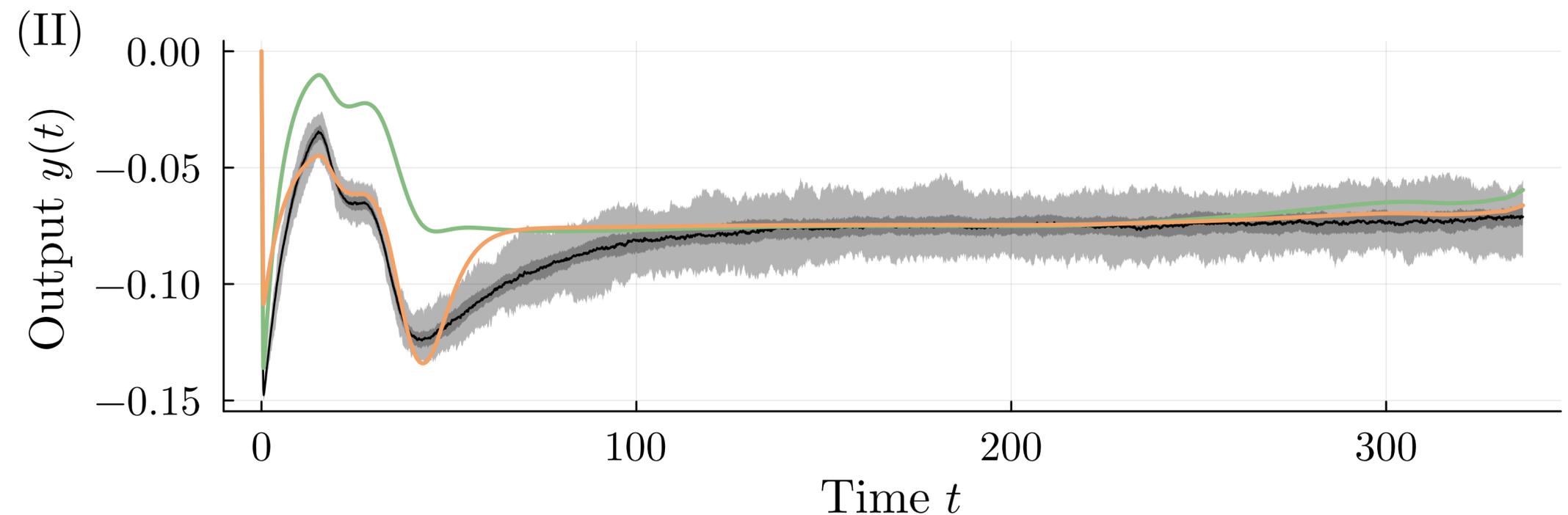
Candidate models:
3-state vs. 4-state

Reference system:
3-state

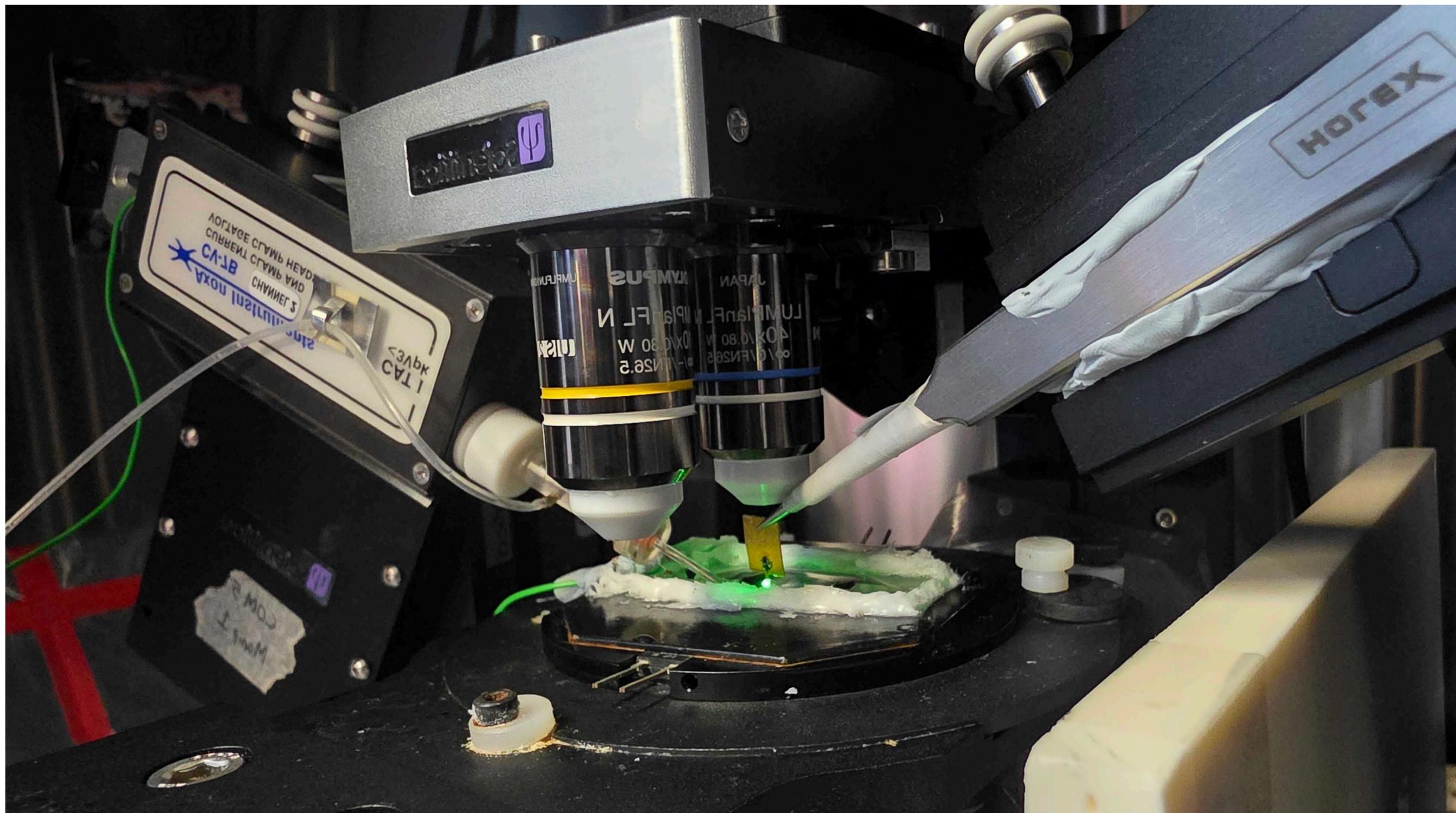


Candidate models:
3-state vs. 4-state

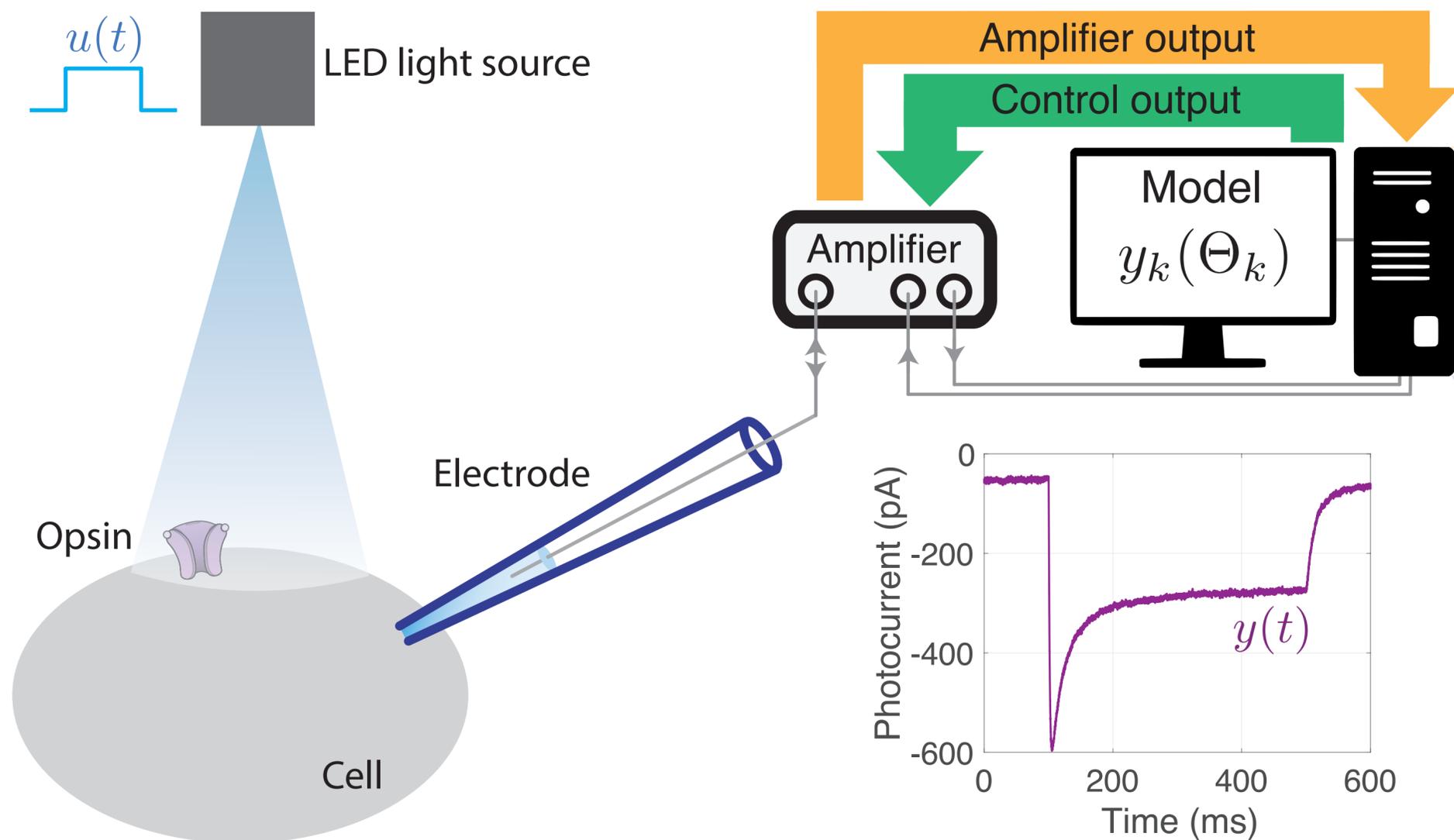
Reference system:
4-state



Experiment



Computer — Experiment



Symphony is an OOP Matlab-based app for electrophysiology recording

~10s

julia

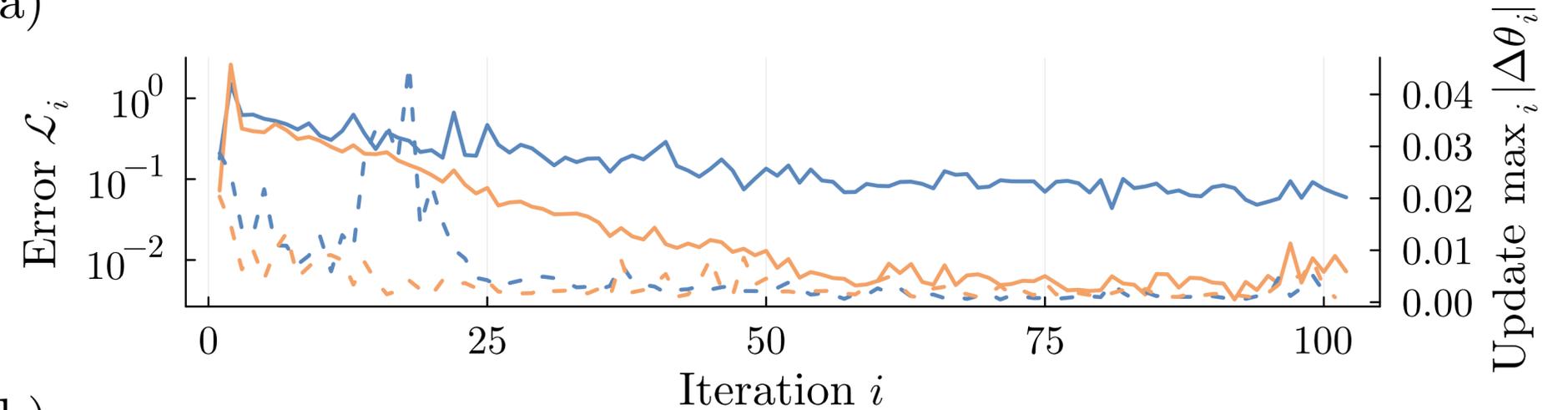
Sockets.jl to open *tcp server* to connect to following each data collection step

~5s

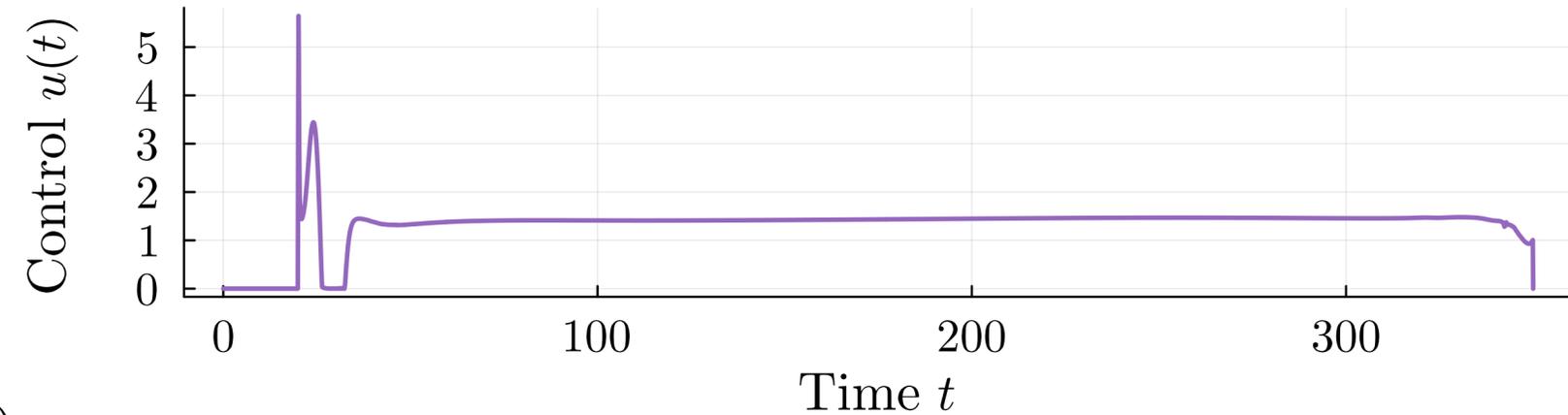
Experiment

Candidate models:
6-state vs. 4-state

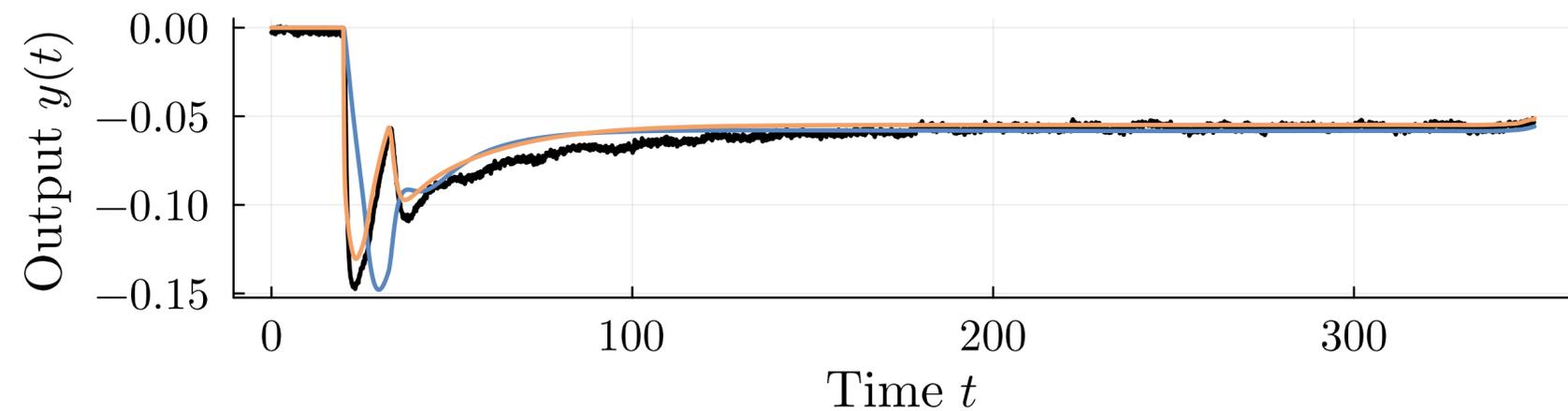
(Ia)



(Ib)

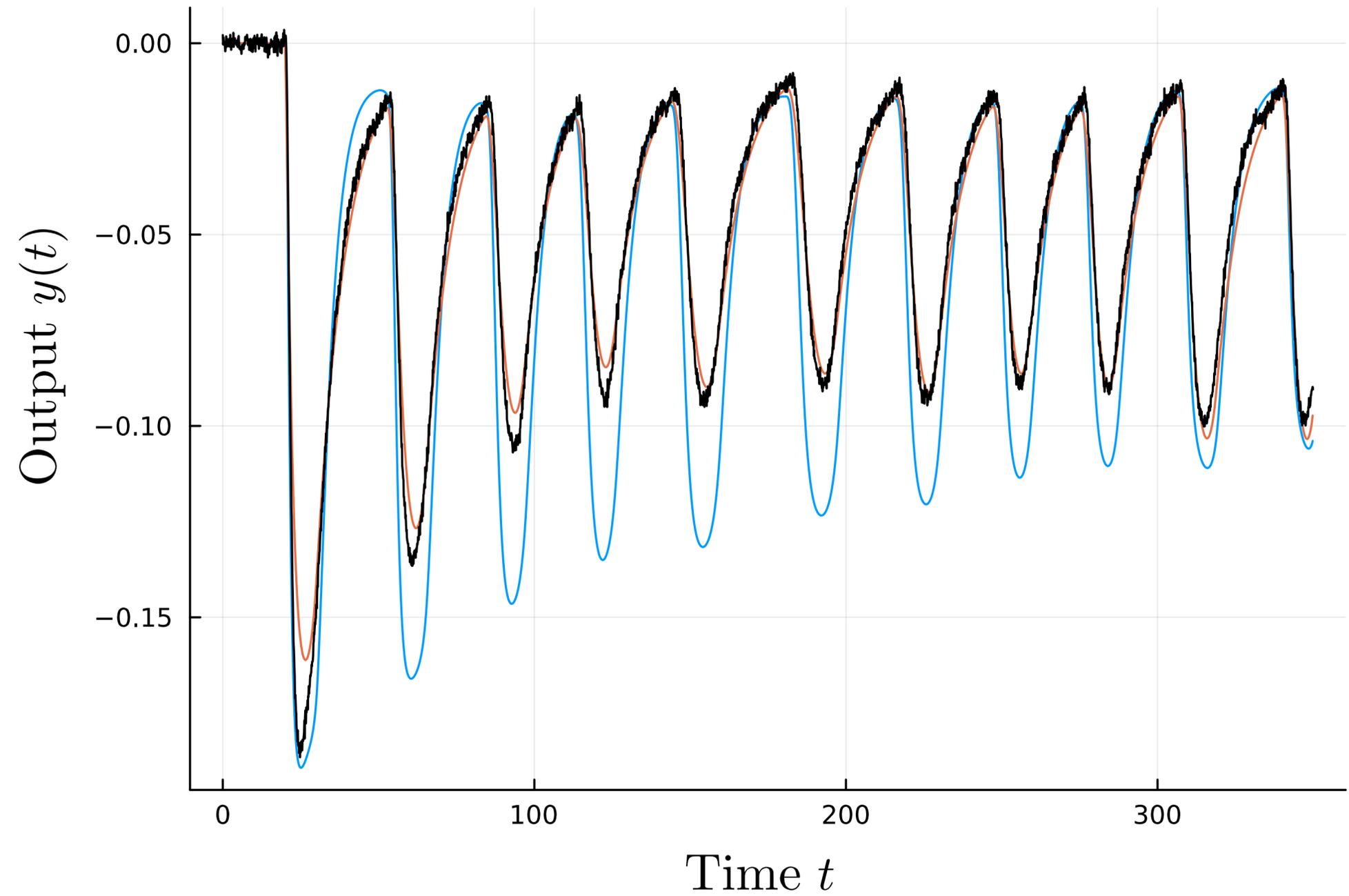


(Ic)



Experiment

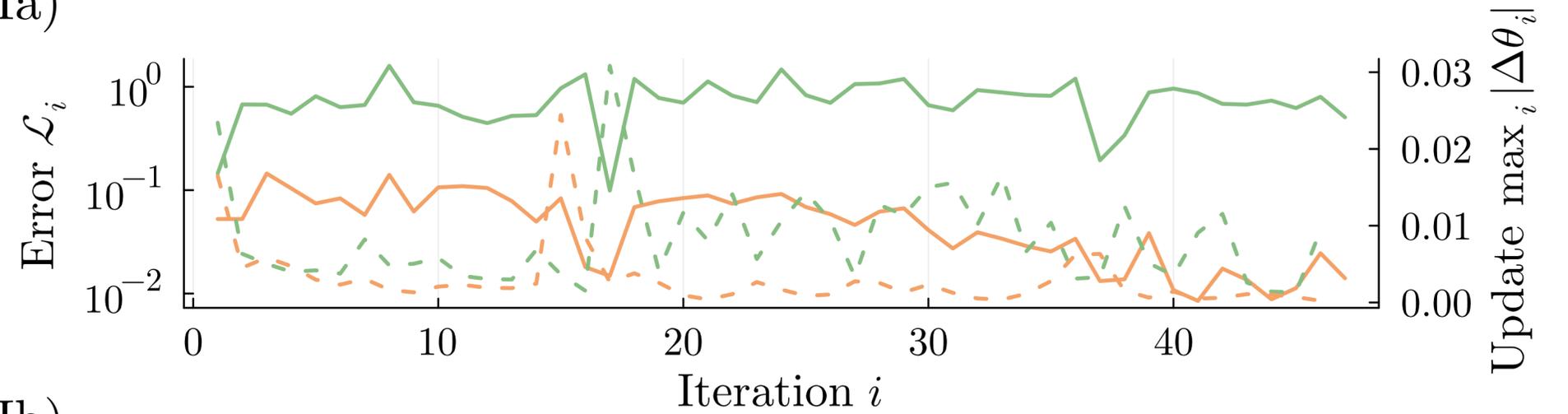
Candidate models:
6-state vs. 4-state



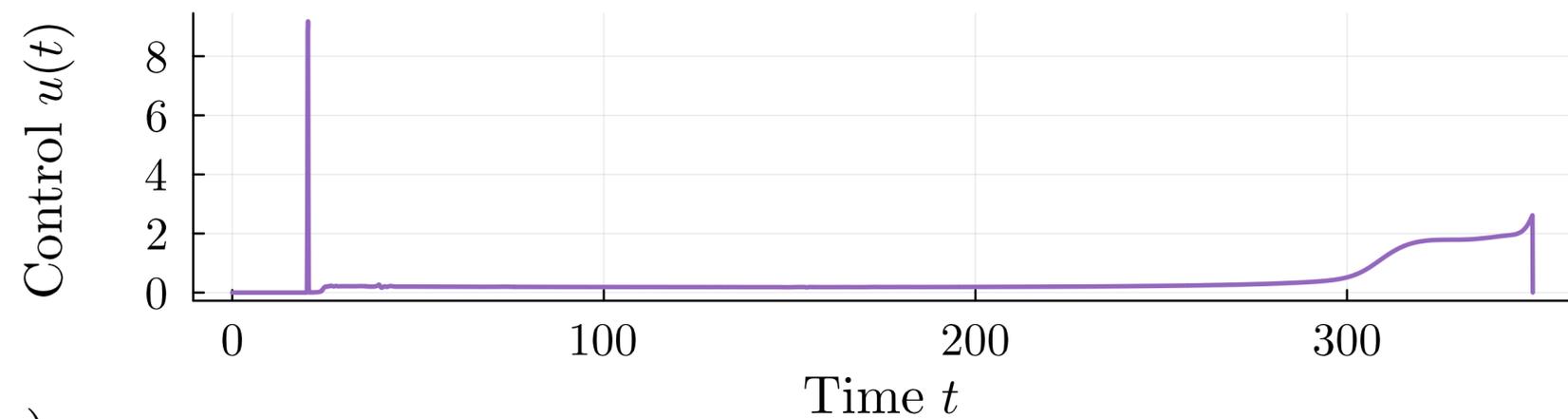
Experiment

Candidate models:
3-state vs. 4-state

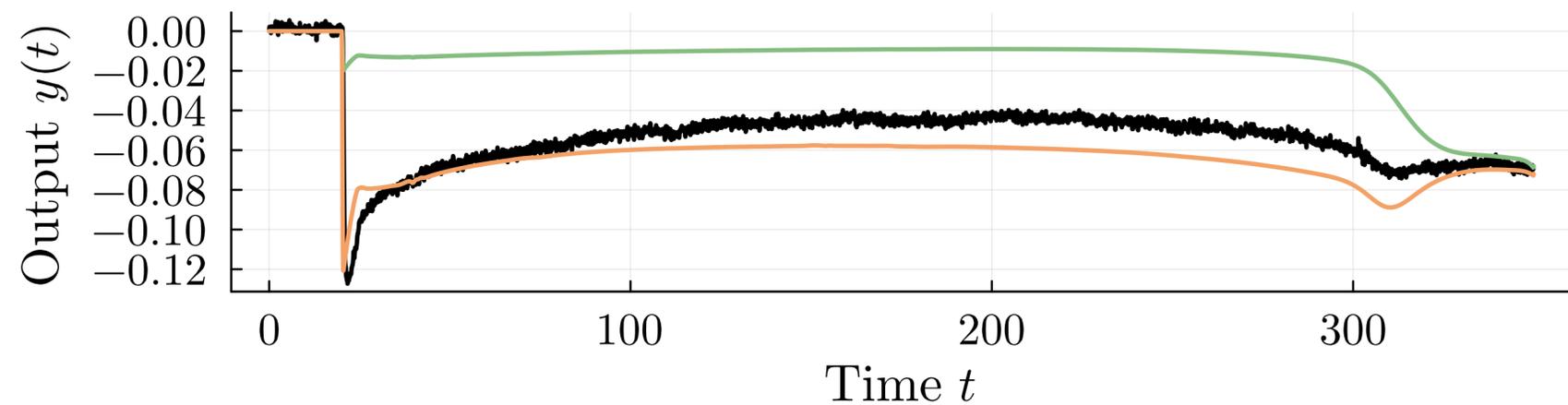
(IIa)



(IIb)



(IIc)



Conclusion

- Closed-loop algorithm for model selection.
- Allowed to select the most accurate opsin model.
- Exploration of the state based on heuristics.

Future work

- Build digital twins.
- Leverage digital twin to optimize the control inputs.
- Apply to other systems.

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