Predicting the response of structurally altered and asymmetrical networks

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MT, Phys. Rev. E ***112****, L042301 (2025)*

Predicting the response of structurally altered and asymmetrical networks …
Optimal network control



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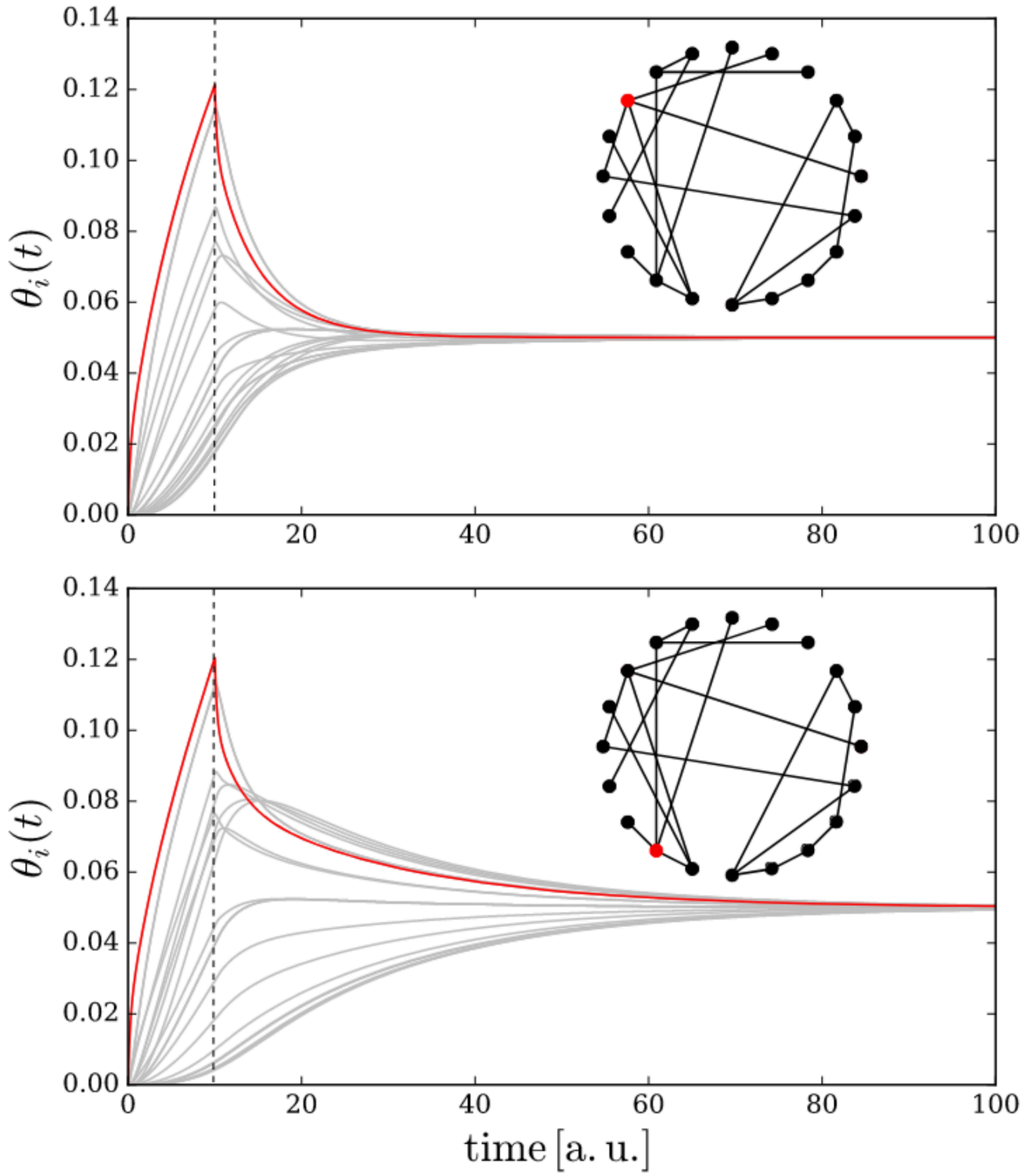
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MT, arXiv:2607.04998 (2026)

Complexity Science, Florida State University — 5/10/2026

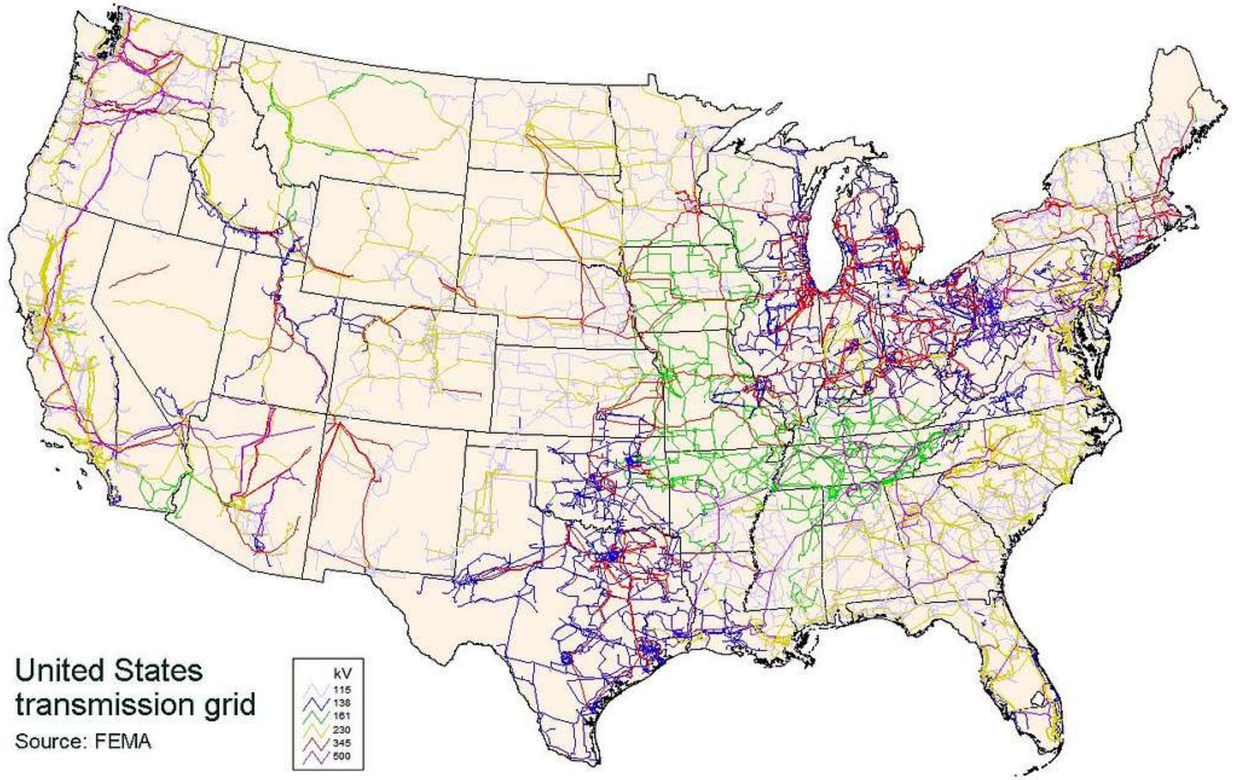
Resilience of networked systems

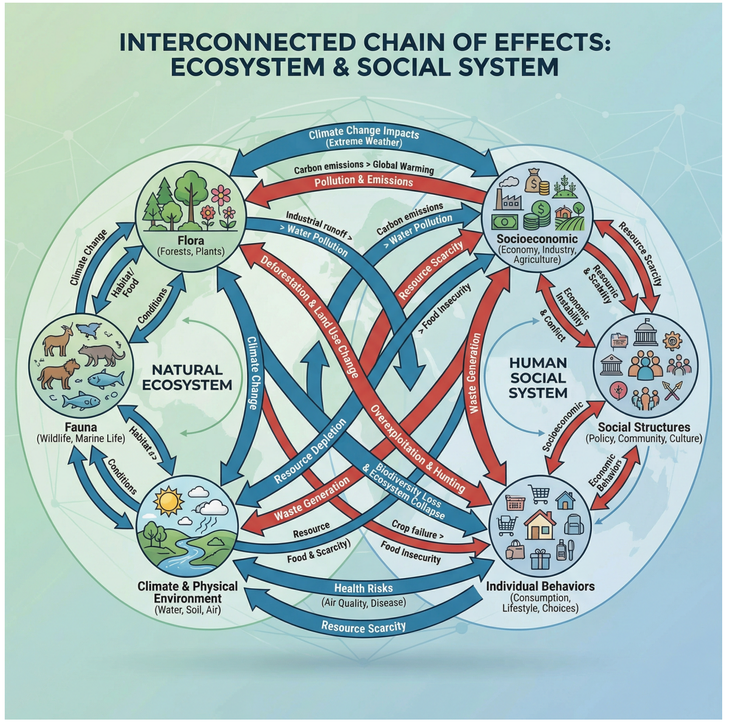
- Dynamics and structure:

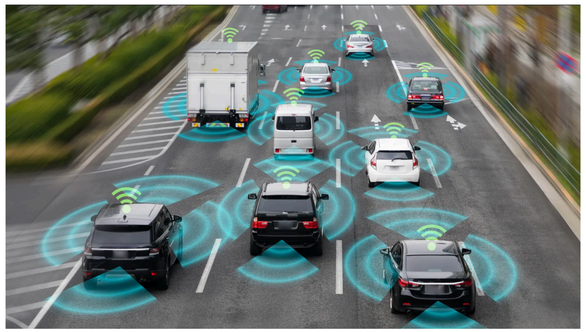


Resilience of networked systems

- Dynamics and structure:



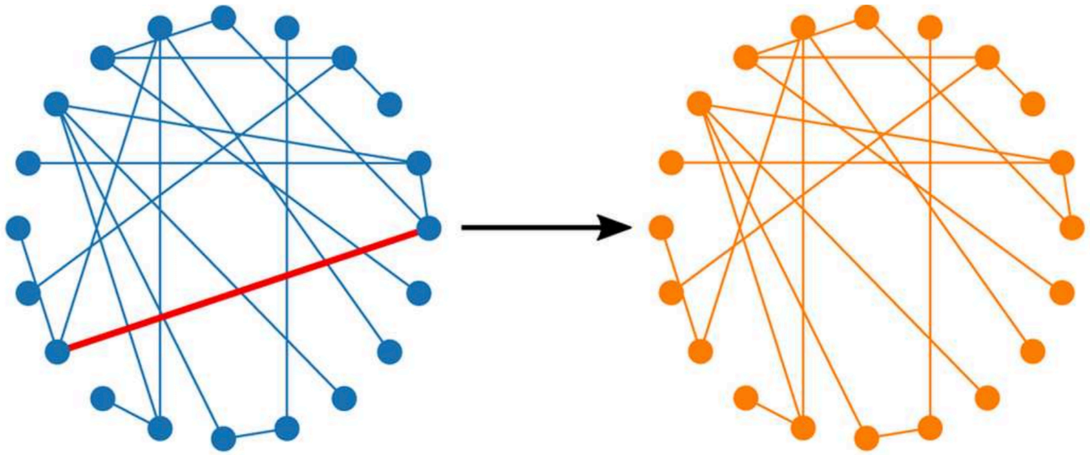




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Resilience of networked systems

- Dynamics and structure:
  - How does the response change when the structure changes?



Resilience of networked systems

- Dynamics and structure:
  - How does the response change when the structure changes?
    - ★ Symmetry of the interaction/perturbation.
    - ★ Normality of the interaction/perturbation.
    - ★ Set of matrices that identify the most impacted nodes.

*MT, Phys. Rev. E **112**, L042301 (2025)*

Networked system

Simple linear system:

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \eta$$

$\mathbf{x} \in \mathbb{R}^N$	State of the system.
$\mathbf{A} \in \mathbb{R}^{N \times N}$	Coupling between the degrees of freedom, assumed symmetric.
$\eta \in \mathbb{R}^N$	External input.

Networked system

Linear approximation of the dynamics around a fixed point:

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \eta \qquad \dot{\mathbf{y}} = \mathbf{F}(\mathbf{y}) + \eta\,, \quad A_{ij} = \frac{\partial F_i}{\partial y_j}\Big|_{\mathbf{y}=\mathbf{y}^*}$$

$\mathbf{x} \in \mathbb{R}^N$	Deviation from the fixed point, i.e. $\mathbf{x} = \mathbf{y} - \mathbf{y}^*$
$\mathbf{A} \in \mathbb{R}^{N \times N}$	Jacobian matrix.
$\eta \in \mathbb{R}^N$	External input.

Networked system

Linear approximation of the dynamics around a fixed point:

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \eta$$

Solution:

$$\mathbf{x}(t) = \int_0^t e^{\mathbf{A}\,(t-t')}\eta(t')\,\mathrm{d}t' \,.$$

Averaged response

Linear approximation of the dynamics around a fixed point:

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \eta$$

Solution:

$$\mathbf{x}(t) = \int_0^t e^{\mathbf{A}\,(t-t')}\eta(t')\,\mathrm{d}t'$$

$$\mathbf{x}(i,t) = [e^{\mathbf{A}t}]_{.,i}$$

$$\eta(i,t) = \hat{\mathbf{e}}_i\,\delta(t)\,,\, i = 1,\ldots,N$$

Averaged response

Linear approximation of the dynamics around a fixed point:

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \eta$$

$$\overline{x_j^2}(t) = \sum_{i=1}^{N} [\mathbf{x}(i,t)\mathbf{x}(i,t)^\top]_{jj} = [e^{\mathbf{A}t} e^{\mathbf{A}^\top t}]_{jj} = [e^{2\mathbf{A}t}]_{jj}$$

Structural perturbation

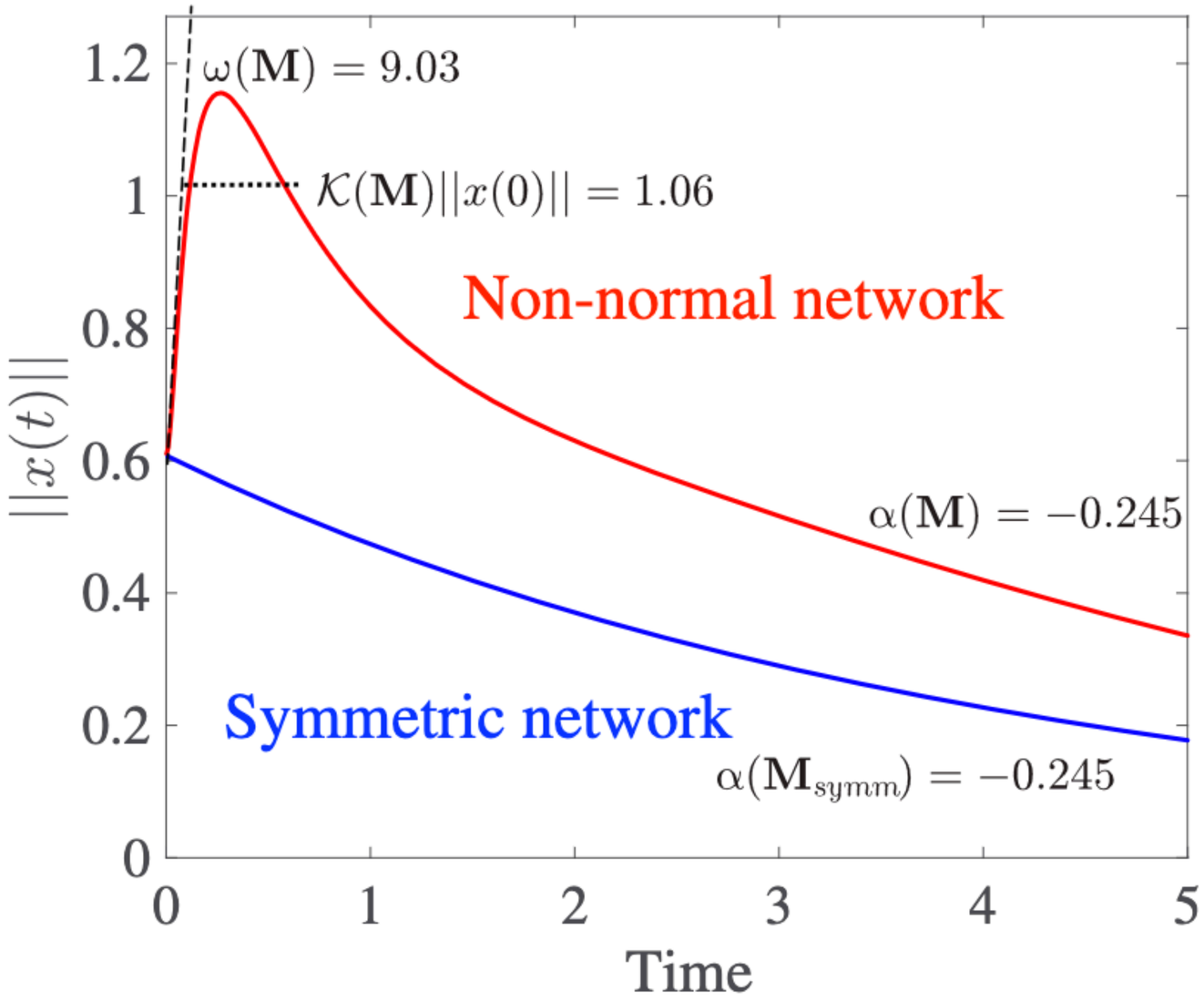
Perturbation of the dynamics:

$$\dot{\mathbf{x}} = (\mathbf{A} + \Delta)\,\mathbf{x} + \eta \qquad \Delta \text{ Structural alteration}$$

$$\overline{x_j^2}(t) = \sum_{i=1}^{N} [\mathbf{x}(i,t)\mathbf{x}(i,t)^\top]_{jj} = [e^{(\mathbf{A}+\boldsymbol{\Delta})t} e^{(\mathbf{A}^\top+\boldsymbol{\Delta}^\top)t}]_{jj}$$

Non-normality $[\mathbf{A} + \boldsymbol{\Delta}, \mathbf{A}^\top + \boldsymbol{\Delta}^\top] \neq 0$

(Non-normality)



M. Asllani et al. *Science advances* 4.12 (2018): eaau9403.

Structural perturbation

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Structural perturbation

Perturbation of the dynamics:

$$\overline{x_j^2}(t) = \sum_{i=1}^{N} [\mathbf{x}(i,t)\mathbf{x}(i,t)^\top]_{jj} = [e^{(\mathbf{A}+\mathbf{\Delta})t} e^{(\mathbf{A}^\top+\mathbf{\Delta}^\top)t}]_{jj}$$

Non-normality $[\mathbf{A}+\mathbf{\Delta}, \mathbf{A}^\top+\mathbf{\Delta}^\top] \neq 0$

$$\overline{x_j^2}(t) = [e^{\mathbf{B}\,t}]_{jj}$$

$$\mathbf{B} = (\mathbf{A}+\Delta)t + (\mathbf{A}+\Delta)^\top t + \frac{1}{2}[\mathbf{A}+\Delta, (\mathbf{A}+\Delta)^\top]t^2 + \frac{1}{12}[\mathbf{A}+\Delta, [\mathbf{A}+\Delta, (\mathbf{A}+\Delta)^\top]]t^3 + \dots$$

Baker–Campbell–Hausdorff formula

Structural perturbation

Perturbation of the dynamics:

$$\overline{x_j^2}(t) = \sum_{i=1}^{N} [\mathbf{x}(i,t)\mathbf{x}(i,t)^\top]_{jj} = [e^{(\mathbf{A}+\mathbf{\Delta})t} e^{(\mathbf{A}^\top+\mathbf{\Delta}^\top)t}]_{jj}$$

Non-normality $[\mathbf{A}+\mathbf{\Delta}, \mathbf{A}^\top+\mathbf{\Delta}^\top] \neq 0$

$$\overline{x_j^2}(t) = [e^{\mathbf{M}_1 t + \mathbf{M_2}\frac{t^2}{2} + \mathbf{M_3}\frac{t^3}{12} + h.o.t.}]_{jj}$$

$$\mathbf{M}_1 = 2\mathbf{A} + \mathbf{\Delta} + \mathbf{\Delta}^\top,$$

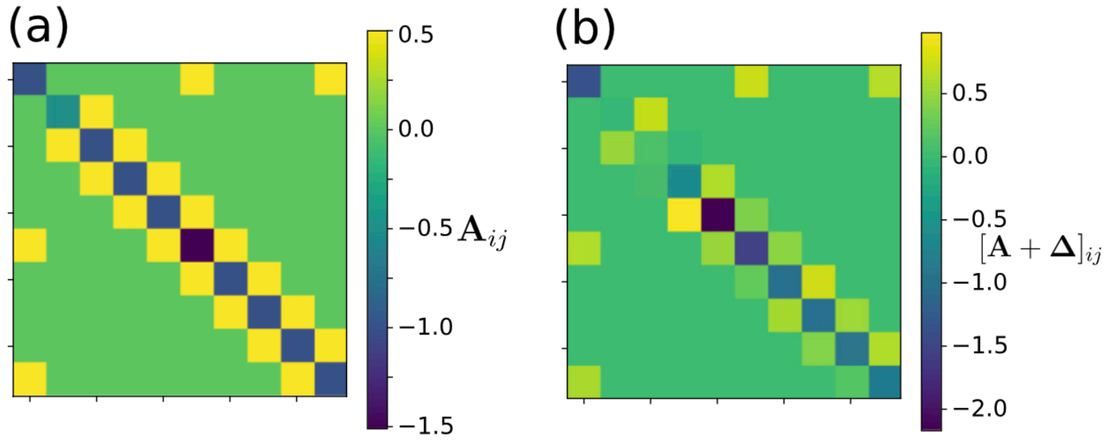
$$\mathbf{M}_2 = [\mathbf{A}, \mathbf{\Delta}^\top - \mathbf{\Delta}] + [\mathbf{\Delta}, \mathbf{\Delta}^\top],$$

$$\mathbf{M}_3 = [\mathbf{\Delta} - \mathbf{\Delta}^\top, [\mathbf{A}, \mathbf{\Delta}^\top - \mathbf{\Delta}] + [\mathbf{\Delta}, \mathbf{\Delta}^\top]\,].$$

Global response

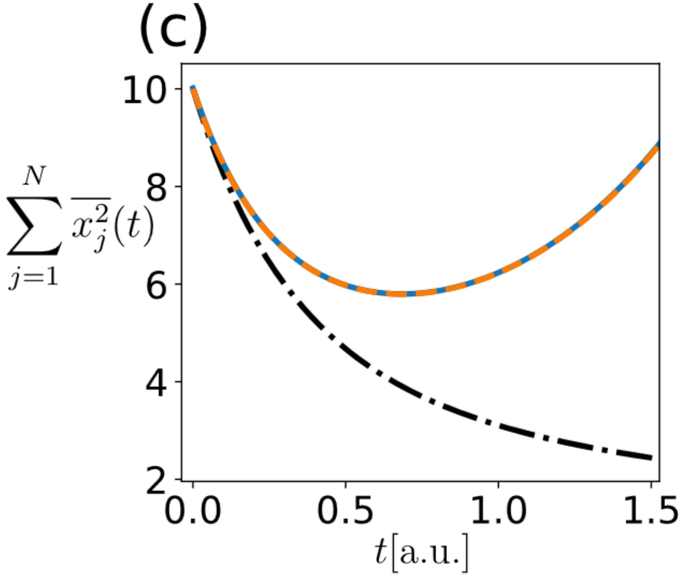
Approximation:

$$\overline{x_j^2}(t) \cong [e^{\mathbf{M}_1 t + \mathbf{M}_2 \frac{t^2}{2} + \mathbf{M}_3 \frac{t^3}{12}}]_{jj}$$



Network level response:

$$\sum_{j=1}^{N} \overline{x_j^2}(t) \cong \mathrm{Tr}\left[e^{\mathbf{M}_1 t + \mathbf{M}_2 \frac{t^2}{2} + \mathbf{M}_3 \frac{t^3}{12}}\right] .$$



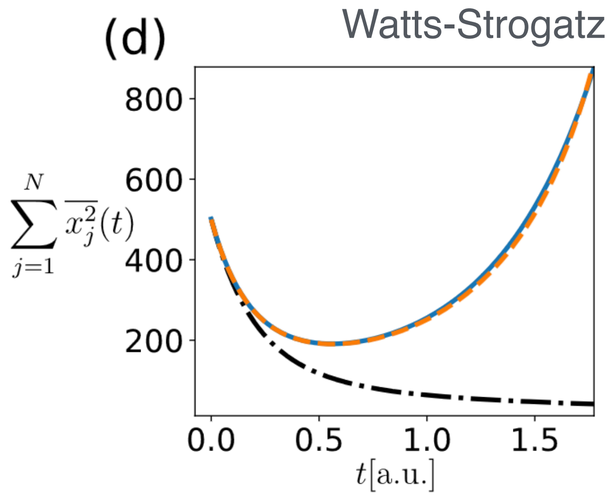
Global response

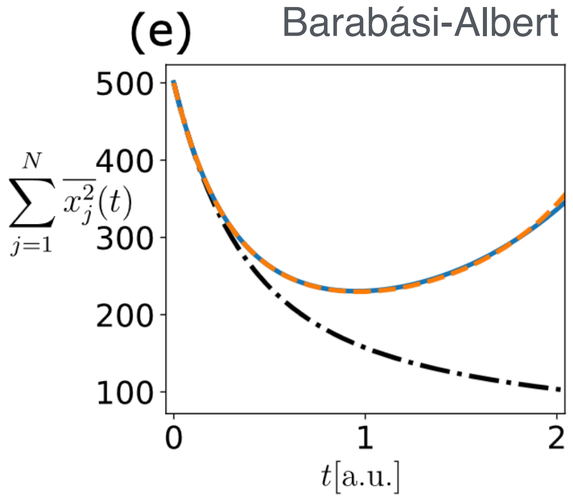
Approximation:

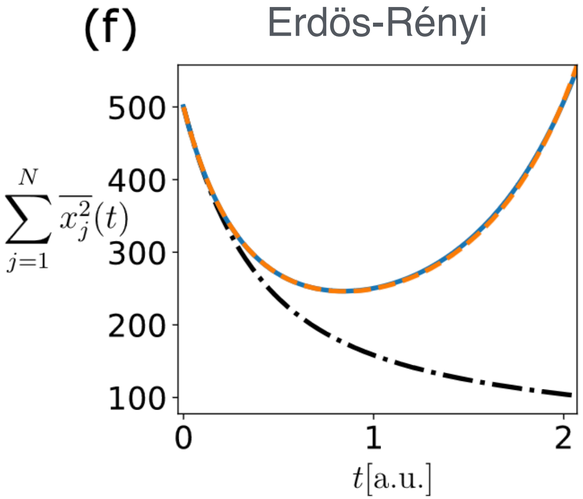
$$\overline{x_j^2}(t) \cong \left[e^{\mathbf{M}_1 t + \mathbf{M_2}\frac{t^2}{2} + \mathbf{M_3}\frac{t^3}{12}}\right]_{jj}$$

Network level response:

$$\sum_{j=1}^{N} \overline{x_j^2}(t) \cong \mathrm{Tr}\left[e^{\mathbf{M}_1 t + \mathbf{M_2}\frac{t^2}{2} + \mathbf{M_3}\frac{t^3}{12}}\right]$$



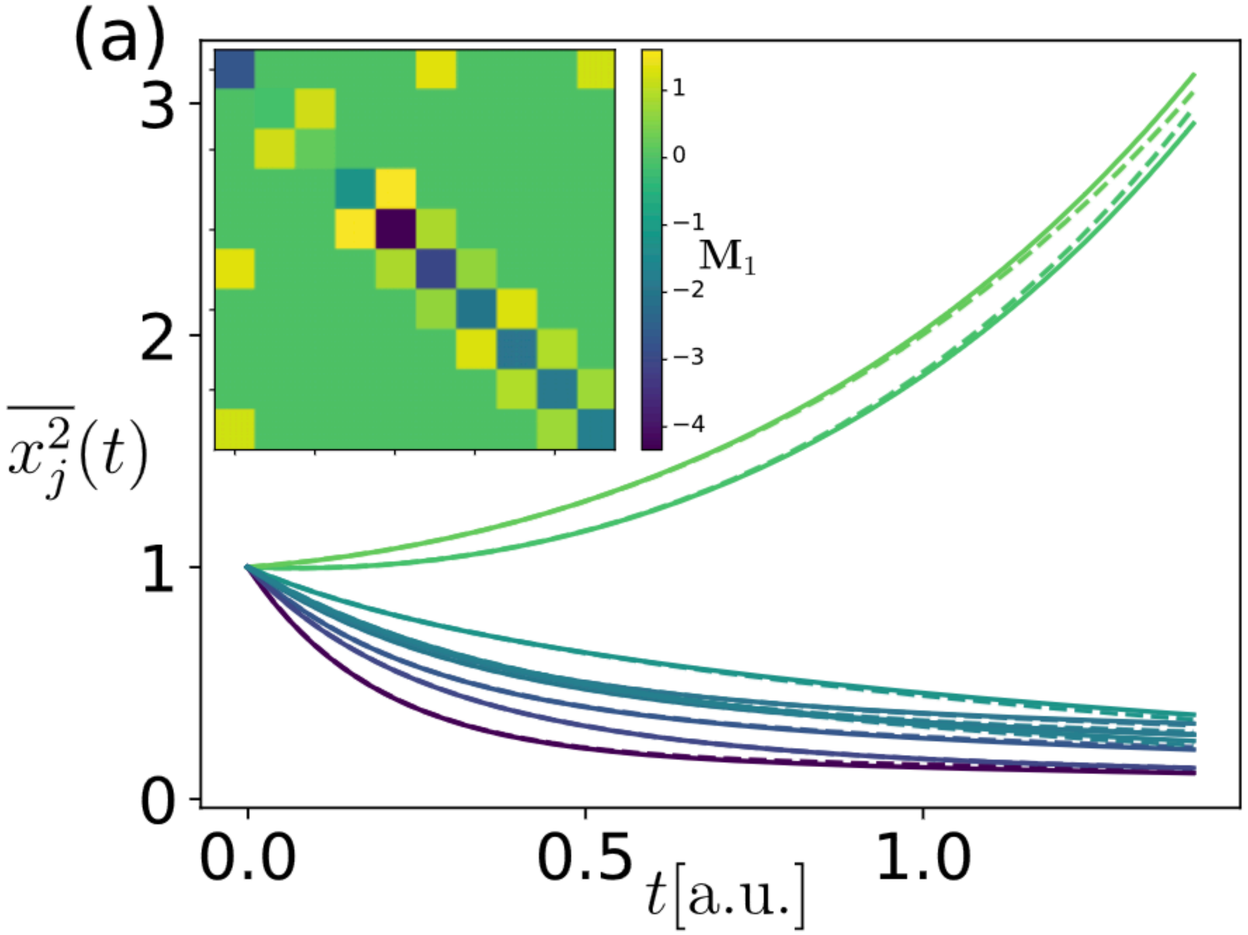




Local response

Node level response:

$$\overline{x_j^2}(t) \cong [e^{\mathbf{M}_1 t + \mathbf{M_2} \frac{t^2}{2} + \mathbf{M_3} \frac{t^3}{12}}]_{jj}$$

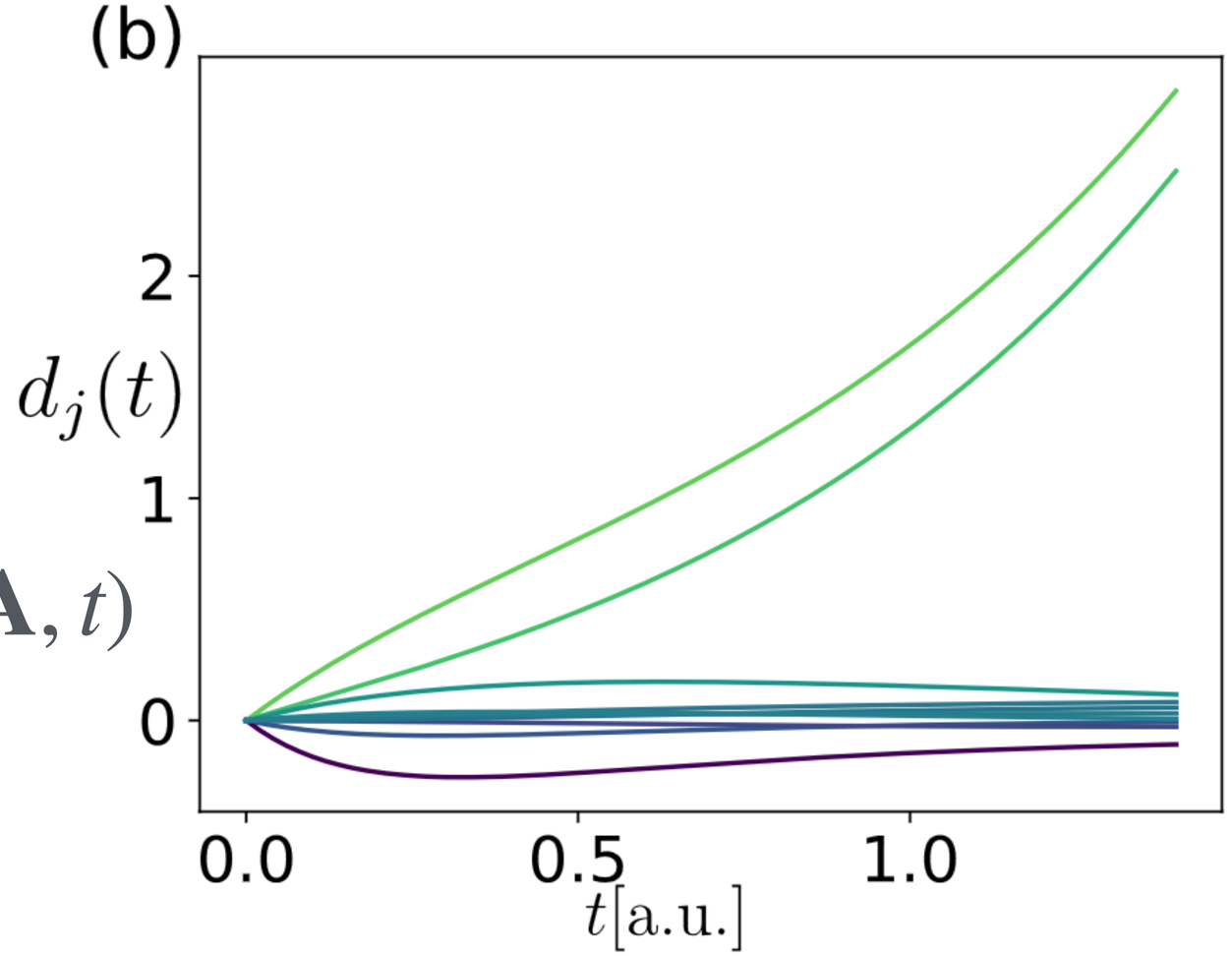


Local response

Node level response:

$$\overline{x_j^2}(t) \cong [e^{\mathbf{M}_1 t + \mathbf{M}_2 \frac{t^2}{2} + \mathbf{M}_3 \frac{t^3}{12}}]_{jj}$$

$$d_j(t) = \overline{x_j^2}(\mathbf{A} + \mathbf{\Delta}, t) - \overline{x_j^2}(\mathbf{A}, t)$$



Global vs. Local response

Network level response:

$$\sum_{j=1}^{N} \overline{x_j^2}(t) \cong N + \mathrm{Tr}[\mathbf{M}_1]\, t + \mathrm{Tr}[\mathbf{M}_1^2]\, \frac{t^2}{2} + \mathrm{Tr}[\mathbf{M}_1^3]\, \frac{t^3}{6}$$

$$+ \left(\mathrm{Tr}[\mathbf{M}_1^4] + 2\,\mathrm{Tr}[\mathbf{M}_1\mathbf{M}_3] + 3\,\mathrm{Tr}[\mathbf{M}_2^2]\right)\, \frac{t^4}{24} + \mathcal{O}(t^5)$$

Node level response:

$$\overline{x_j^2}(t) \cong 1 + [\mathbf{M}_1]_{jj}\, t + \left([\mathbf{M}_1^2]_{jj} + [\mathbf{M}_2]_{jj}\right)\, \frac{t^2}{2}$$

$$+ \left([\mathbf{M}_1^3]_{jj} + 3[\mathbf{M}_1\mathbf{M}_2]_{jj} + \frac{1}{2}[\mathbf{M}_3]_{jj}\right)\, \frac{t^3}{6} + \mathcal{O}(t^4)$$

Global vs. Local response

Network level

$$\sum_{j=1}^{N} \overline{x_j^2}(t) \cong N + \mathrm{Tr}[\mathbf{M}_1]\, t + \mathrm{Tr}[\mathbf{M}_1^2]\, \frac{t^2}{2} + \mathrm{Tr}[\mathbf{M}_1^3]\, \frac{t^3}{6}$$

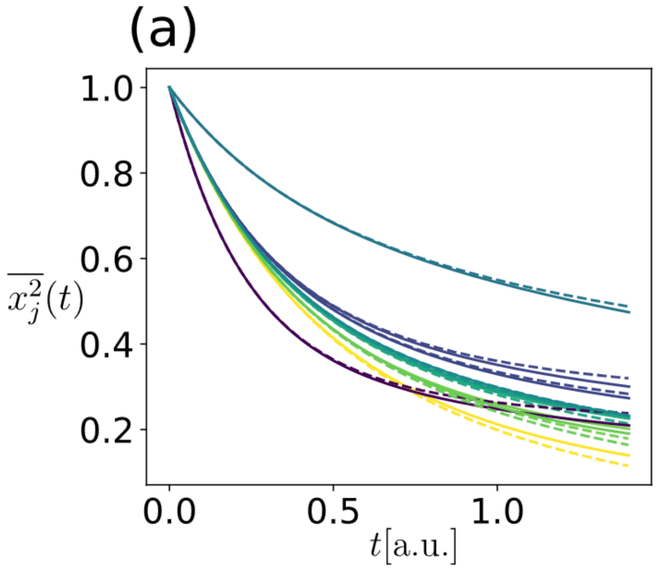
$$+ \left(\mathrm{Tr}[\mathbf{M}_1^4] + 2\,\mathrm{Tr}[\mathbf{M}_1\mathbf{M}_3] + 3\,\mathrm{Tr}[\mathbf{M}_2^2]\right)\, \frac{t^4}{24} + \mathcal{O}(t^5)$$

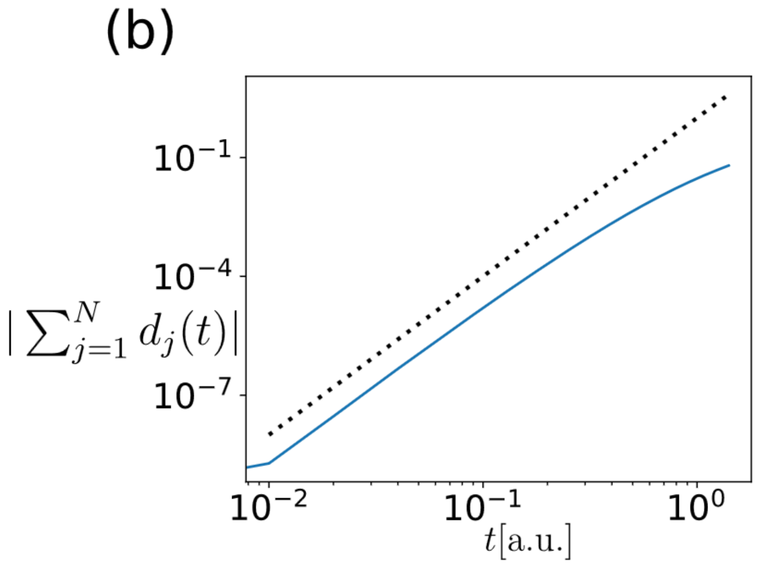
Node level

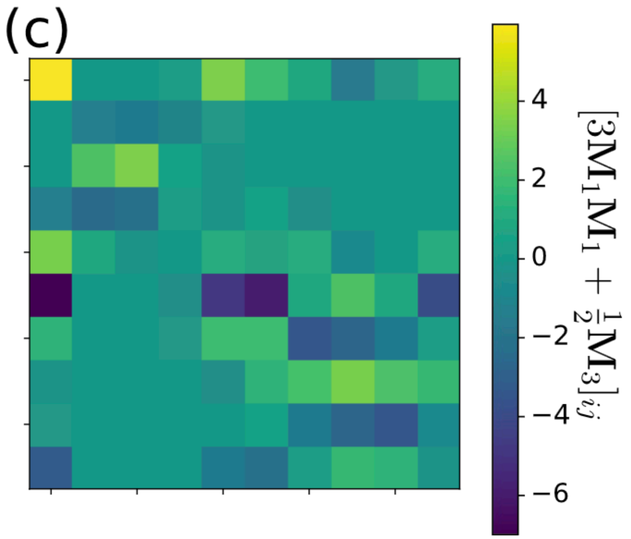
$$\overline{x_j^2}(t) \cong 1 + [\mathbf{M}_1]_{jj}\, t + \left([\mathbf{M}_1^2]_{jj} + [\mathbf{M}_2]_{jj}\right)\, \frac{t^2}{2}$$

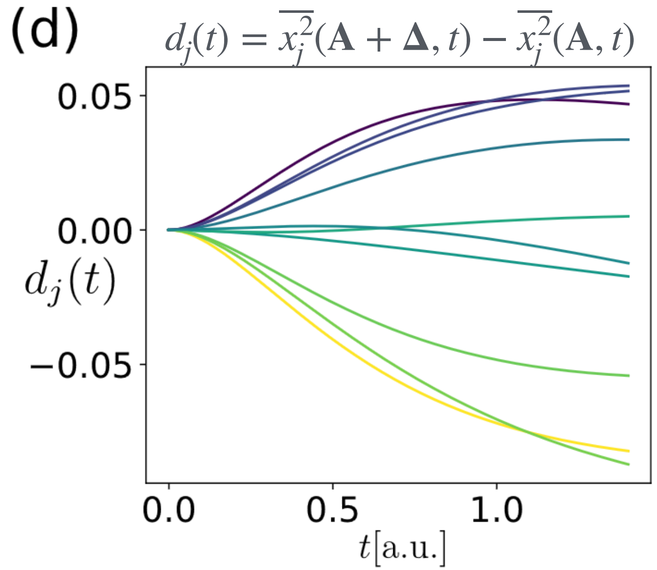
$$+ \left([\mathbf{M}_1^3]_{jj} + 3[\mathbf{M}_1\mathbf{M}_2]_{jj} + \frac{1}{2}[\mathbf{M}_3]_{jj}\right)\, \frac{t^3}{6} + \mathcal{O}(t^4)$$

Skew-symmetric perturbation









Conclusion: Part 1

- Set of matrices to identify impacted nodes.
- Perturbations not visible in global response but visible locally.

$$\overline{x_j^2}(t) = \left[e^{\mathbf{M}_1 t + \mathbf{M_2}\frac{t^2}{2} + \mathbf{M_3}\frac{t^3}{12} + h.o.t.}\right]_{jj}$$

$$\mathbf{M}_1 = 2\mathbf{A} + \boldsymbol{\Delta} + \boldsymbol{\Delta}^\mathsf{T},$$

$$\mathbf{M}_2 = [\mathbf{A}, \boldsymbol{\Delta}^\mathsf{T} - \boldsymbol{\Delta}] + [\boldsymbol{\Delta}, \boldsymbol{\Delta}^\mathsf{T}],$$

$$\mathbf{M}_3 = [\boldsymbol{\Delta} - \boldsymbol{\Delta}^\mathsf{T}, [\mathbf{A}, \boldsymbol{\Delta}^\mathsf{T} - \boldsymbol{\Delta}] + [\boldsymbol{\Delta}, \boldsymbol{\Delta}^\mathsf{T}]\,].$$

Optimal control of the interaction

Interesting questions:

- By controlling the interaction between the neurons, can we control the activity?

$$\dot{\mathbf{x}}_i = \mathbf{F}_i(\mathbf{x}_i) + \sum_{j=1}^{N} W_{ij}\mathbf{H}_i(\mathbf{x}_i, \mathbf{x}_j) + u_i(t)\,, i = 1,...N$$

Node - Interaction

$$\dot{\mathbf{x}}_i = \mathbf{F}_i(\mathbf{x}_i) + \sum_{j=1}^{N} W_{ij}(t)\mathbf{H}_i(\mathbf{x}_i, \mathbf{x}_j)\,, i = 1,...N$$

MT, arXiv:2607.04998 (2026)

Optimal control of the interaction

Simple linear time variant model:

$$\dot{x}_i = \alpha\,(x_i^* - x_i) + \sum_{j=1}^{N} W_{ij}(t) x_j\,, i = 1,...N\,,$$

Network control:

$$W_{ij}(t) = U_{ij}(t) \qquad i,j = 1,...N$$

Precision and control cost:

$$\mathscr{C}_e^* = \frac{1}{2}\int_0^T \sum_{j=1}^{N} (x_j - \tilde{x}_j)^2\,\mathrm{d}t\,, \quad \mathscr{C}_u = \frac{1}{2}\int_0^T \sum_{i,j=1}^{N} U_{ij}^2(t)\,\mathrm{d}t\,.$$

Optimal control of the interaction

Lagrangian function: $$\mathscr{L}(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{U}) = \int_0^T c_u(t) + c_e(t) + \lambda^\top \mathbf{g}(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{U})\, \mathrm{d}t' .$$

$$g_i(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{U}) = \dot{x}_i - \alpha(x_i^* - x_i) - \sum_{j=1}^{N} U_{ij}(t) x_j \quad i = 1,...N .$$

Adjoint state dynamics: $$\dot{\lambda}^\top = \mathbf{p}(\mathbf{x}, \dot{\mathbf{x}}) + \lambda^\top \frac{\partial \mathbf{g}}{\partial \mathbf{x}} .$$

$$\lambda(T) = 0.$$

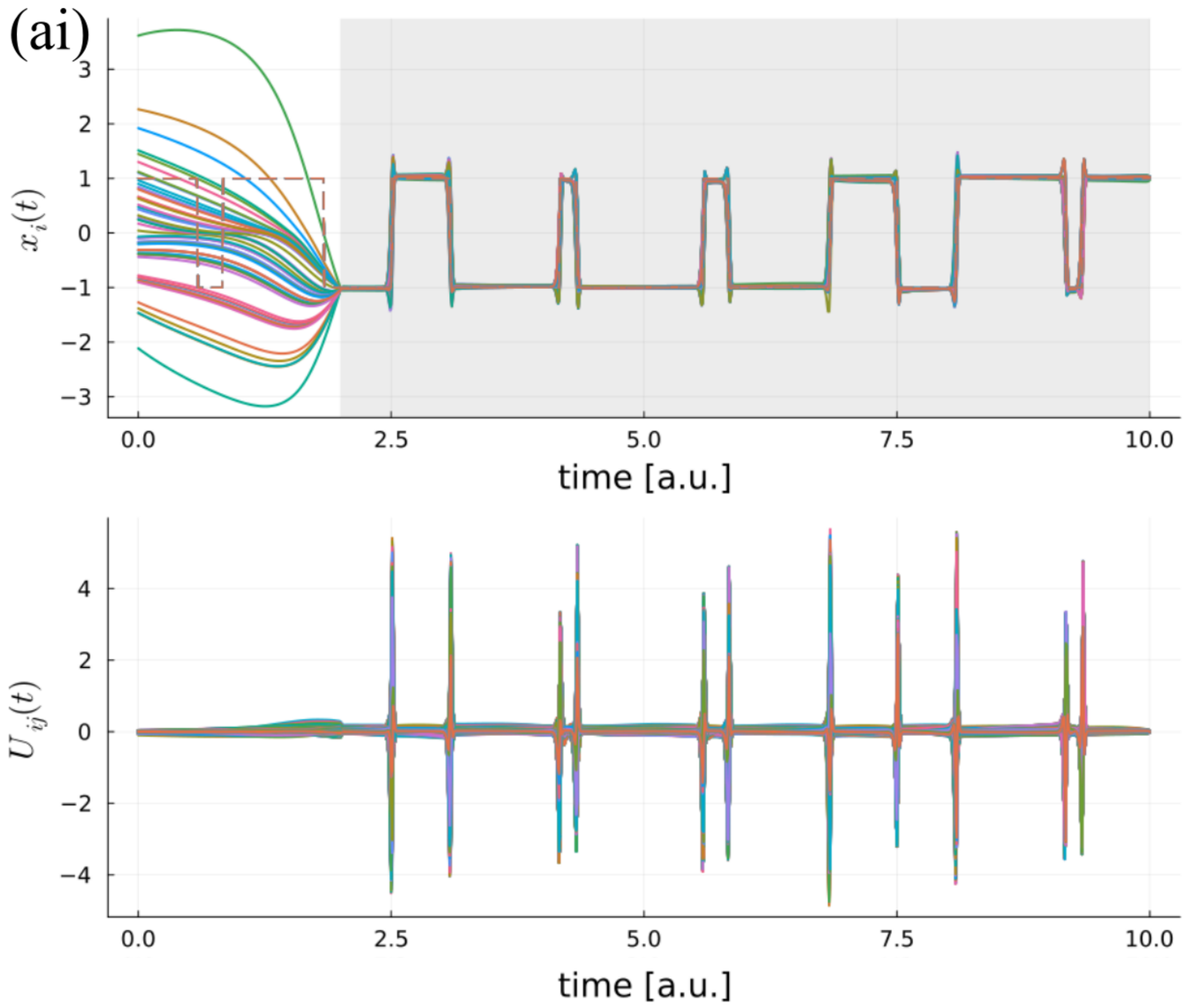
$$\frac{\delta \mathscr{C}_e}{\delta U_{ij}} = \mathbf{p}(\mathbf{x}, \dot{\mathbf{x}}) \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}U_{ij}} .$$

Derivative: $$\frac{\delta \mathscr{L}}{\delta U_{ij}} = \frac{\partial c_u}{\partial U_{ij}} - \lambda_i x_j .$$

MT, arXiv:2607.04998 (2026)

-Controlling only the pair-wise interaction;
-N=50 nodes;
-x* normally distributed.
-Target signal:

$$\tilde{x}_i(t) = \mathrm{sign}[\cos(2\pi 0.3\, t)\cos(2\pi 0.4\, t + 0.1)]\,.$$

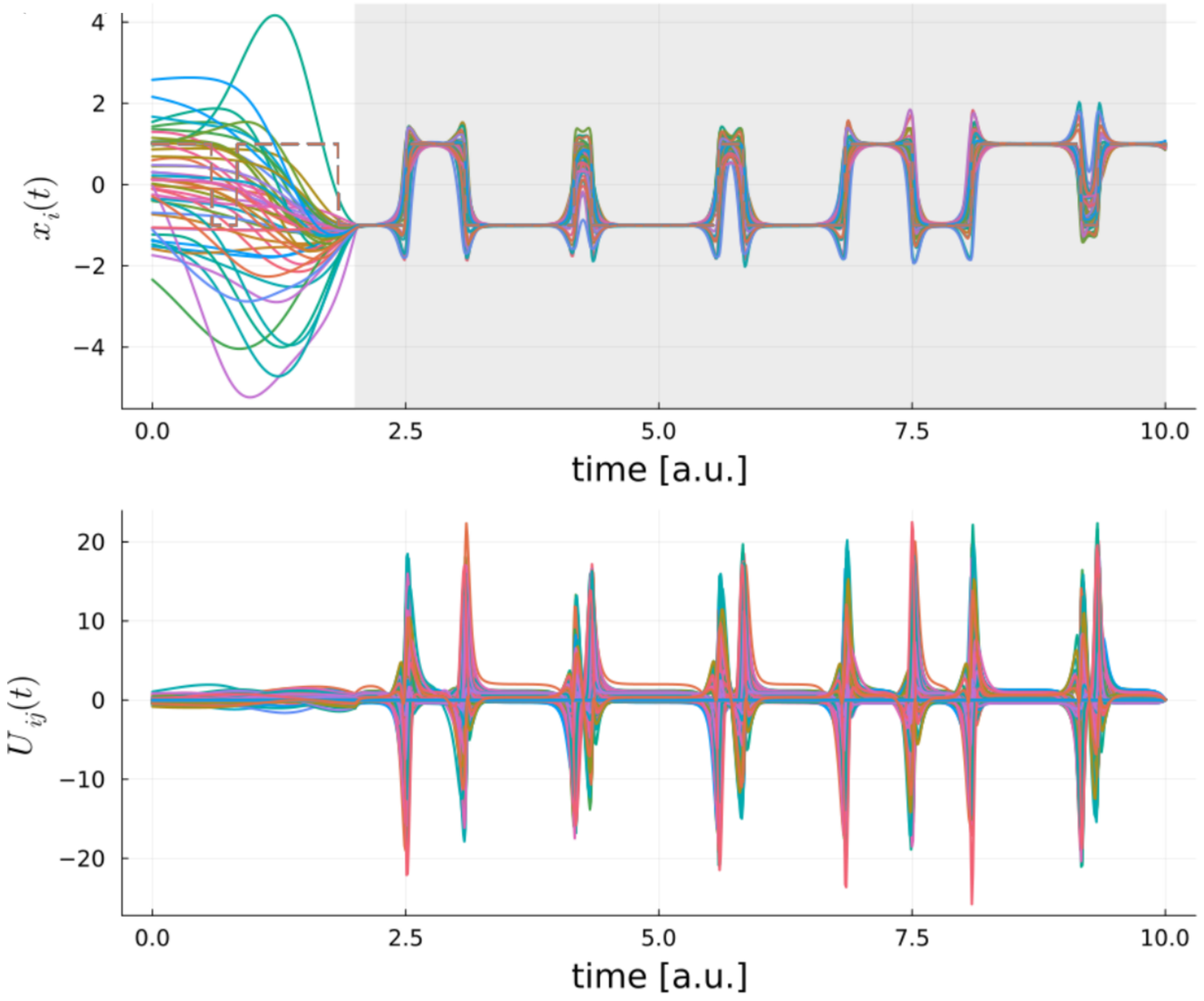


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Only 8% of the interaction controlled.



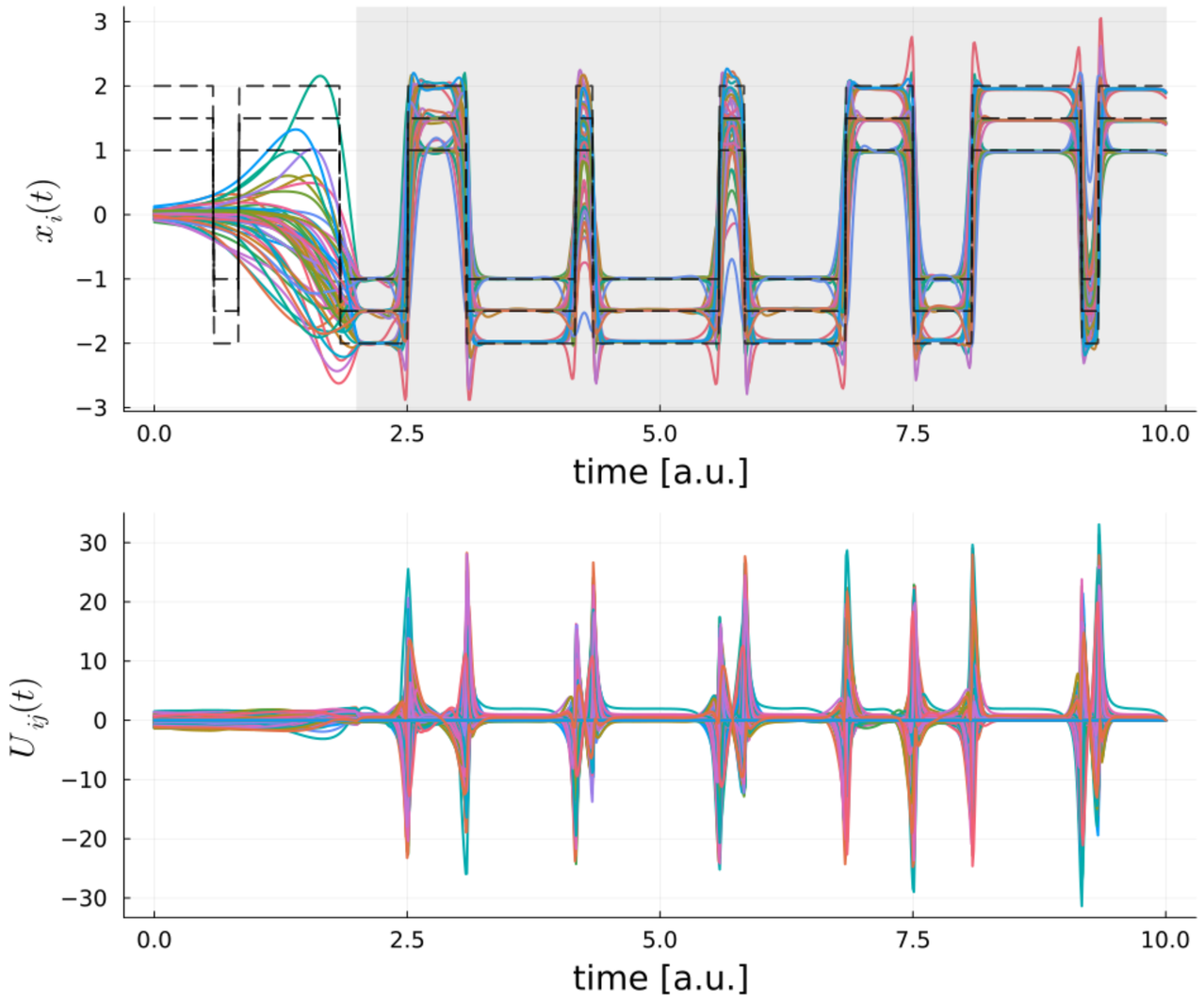
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-Controlling only the pair-wise interaction;
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-x* normally distributed.
-Target signal:

$$\tilde{x}_i(t) = a_i \,\mathrm{sign}[\cos(2\pi 0.3\, t)\cos(2\pi 0.4\, t + 0.1)]\,.$$

$$a_i \in \{1, 1.5, 2\}$$

Only 8% of the interaction controlled.



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Optimal control of the interaction

Population of neurons:

$$\tau \dot{r}_i = \frac{\Delta}{\pi\tau} + 2 r_i v_i \,,$$

$$\tau \dot{v}_i = \eta + v_i^2 - (\pi\tau r_i)^2 + \tau \sum_{j=1}^{N} W_{ij}(t) r_j \,,$$

Network control:

$$W_{ij}(t) = U_{ij}(t) \qquad i,j = 1,...N$$

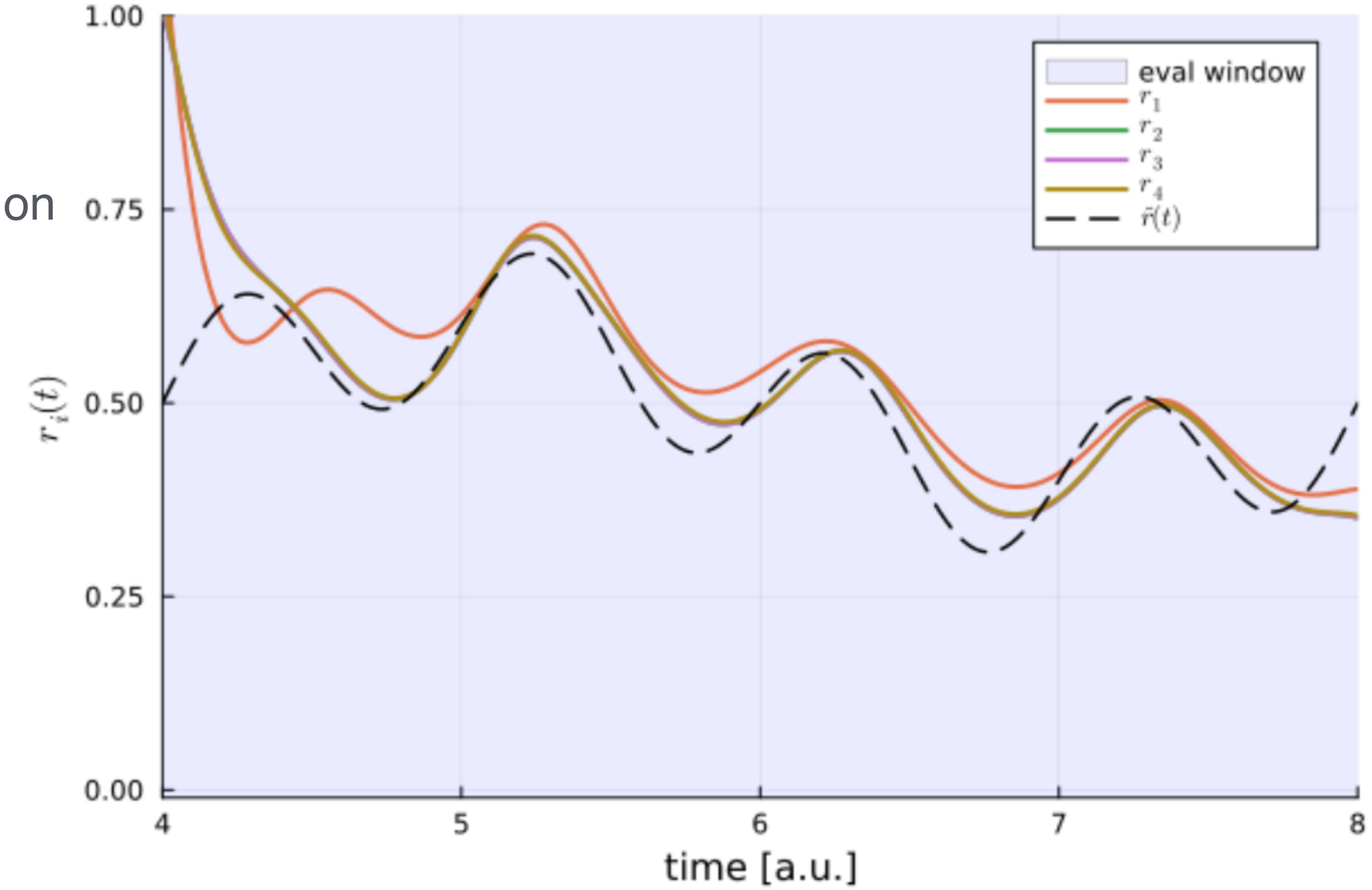
Precision and control cost:

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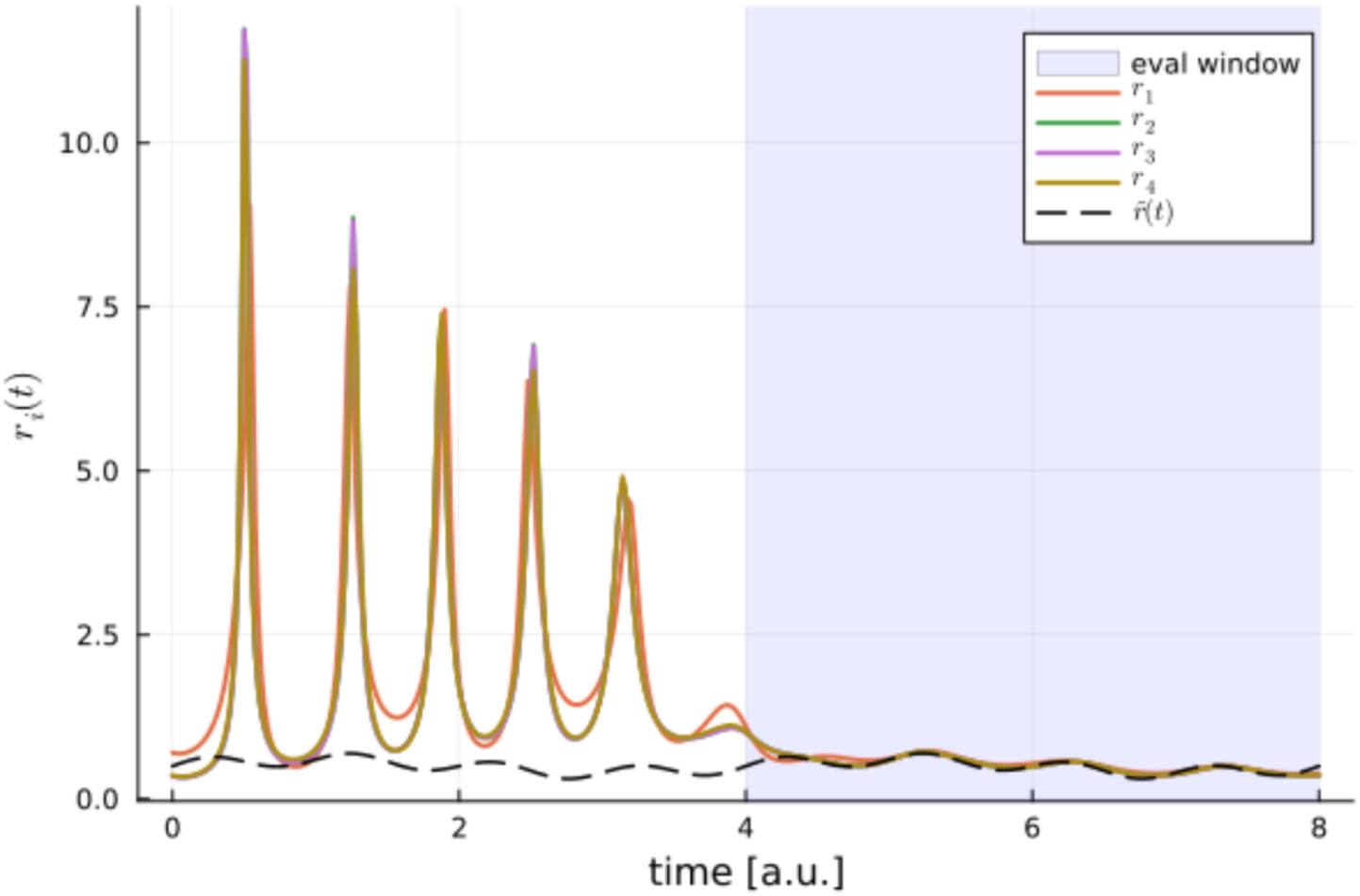
Optimal control of the interaction

-Controlling the interaction, bigger cost on self-interaction;
-N=4 populations of neurons;



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Optimal control of the interaction



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Conclusion: Part 2

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- Network control of the activity.
- Control of neurons with artificial connections (e.g. opsin).

Future work

- Include synaptic plasticity.
- Account for realistic contraints.
- Control the network by controlling the nodes.

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